Exercises on Control Theory

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Exercise 1. Solve the following control problem

$$\max \int_0^1 (uy - u^2 - 2y^2)\, dx, \qquad \text{subject to} \quad \dot{y} = y + u, \;\; y(0) = 3,$$

where $y = y(x)$ and $u = u(x)$.

Solution. The Hamiltonian $H(x, y, u, p)$, where $p = p(x)$, of the problem is

$$H(x, y, u, p) = uy - u^2 - 2y^2 + p(y + u),$$

from which the optimum conditions $\partial H/\partial u = 0, \dot{p} = -\partial H/\partial y$ give

$$y - 2u + p = 0 \qquad \text{and} \qquad \dot{p} = -u + 4y - p.$$

From the first equation we get $u = (y + p)/2$, from which, by substitution into $\dot{p} = -u + 4y - p$, the equation $\dot{p} = -(y + p)/2 + 4y - p$ follows.

We then write the dynamical system

$$\begin{pmatrix} \dot{p} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3/2 & 7/2 \\ 1/2 & 3/2 \end{pmatrix} \begin{pmatrix} p \\ y \end{pmatrix}, \qquad \text{with} \quad A = \begin{pmatrix} -3/2 & 7/2 \\ 1/2 & 3/2 \end{pmatrix}.$$

Since the eigenvalues of the constant matrix A are $\lambda = 2, -2$ and the associated eigenvectors are $\boldsymbol{v}_1 = (1, 1)$ and $\boldsymbol{v}_2 = (7, -1)$, respectively, we can conclude that the matrix A is diagonalizable and write the solution of the dynamical system in the form

$$\begin{pmatrix} p \\ y \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2x} + C_2 \begin{pmatrix} 7 \\ -1 \end{pmatrix} e^{-2x},$$

that is by components

$$\begin{cases} p(x) = C_1 e^{2x} + 7C_2 e^{-2x} \\ y(x) = C_1 e^{2x} - C_2 e^{-2x}. \end{cases}$$

The initial condition $y(0) = 3$ and the trasversality condition $p(1) = 0$ give the algebraic linear system $C_1 - C_2 = 3$ and $C_1 e^2 + 7C_2 e^{-2} = 0$, from which we obtain the values of the constants $C_1 = 21/(7 + e^4)$ and $C_2 = -3e^4/(7 + e^4)$ and finally the solution of the control problem

$$\begin{cases} p(x) = \dfrac{21}{7 + e^4}\, e^{2x} - \dfrac{21e^4}{7 + e^4}\, e^{-2x} \\[2ex] y(x) = \dfrac{21}{7 + e^4}\, e^{2x} + \dfrac{3e^4}{7 + e^4}\, e^{-2x}. \end{cases}$$

with $u(x) = (y + p)/2$ given by

$$u(x) = \frac{21}{7 + e^4}\, e^{2x} - \frac{9e^4}{7 + e^4}\, e^{-2x}.$$