Linear Dynamical Systems with Constant Coefficients
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A linear dynamical system with constant coefficients in $\mathbb{R}^n$ is given by

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \vdots \\ \dot{y}_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ \vdots \\ y_n(0) \end{pmatrix} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \\ \vdots \\ \bar{y}_n \end{pmatrix}, \tag{1}$$

where $y_i = y_i(x)$, $\dot{y}_i$ denotes the derivative $\dot{y}_i = dy_i(x)/dx$, the a_{hk}'s are real numbers and the n conditions $y_i(0) = \bar{y}_i \in \mathbb{R}$ are the initial conditions. If we set

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

and $\boldsymbol{y} = (y_1, y_2, y_3, \cdots, y_n)$, we can rewrite the system in the more compact form $\dot{\boldsymbol{y}} = A\boldsymbol{y}$, such that the pair $\dot{\boldsymbol{y}} = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \bar{\boldsymbol{y}}$ is a *Cauchy problem* and the technique to solve it involves the spectral properties of the constant matrix A. In particular, we distinguish between the case of diagonalizable matrix A, in which we show two methods, and the case of non diagonalizable matrix A, in which we show one of two methods, only, because the other one involves *Jordan's canonical form*, that we neglect for the sake of semplicity.

1 Case of diagonalizable matrix

First method. If the matrix A is diagonalizable, that is if there exists a basis of the vecotr space $\mathbb{R}^n$ formed by the n eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \cdots, \boldsymbol{v}_n$, corresponding to the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$ of the matrix A, then the solution of the system (1) is given by

$$\begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ \vdots \\ y_n(x) \end{pmatrix} = C_1 \boldsymbol{v}_1 \, e^{\lambda_1 x} + C_2 \boldsymbol{v}_2 \, e^{\lambda_2 x} + C_3 \boldsymbol{v}_3 \, e^{\lambda_3 x} + \cdots + C_n \boldsymbol{v}_n \, e^{\lambda_n x}, \tag{2}$$

where the coefficients $C_1, C_2, C_3, \ldots, C_n$ are real numbers which can be calculated by imposing the initial conditions $\boldsymbol{y}(0) = \bar{\boldsymbol{y}}$. It is straightforward to verify that the vector given in (2) is solution of the system (1) written in the form $\dot{\boldsymbol{y}} = A\boldsymbol{y}$. We recall that for a complex eigenvalue $\lambda = a + ib$ it can be useful to use *Euler's formula* $e^{i\theta} = \cos\theta + i \sin\theta$.

Second method. After the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$ of the matrix A with their algebraic multiplicities have been calculated, one writes each $y_i(x)$ in the form of linear combination of all the exponentials $e^{\lambda_k x}$, for each real eigenvalue λ_k, considered only once, and of all the terms $e^{\alpha x} \cos \beta x$, $e^{\alpha x} \sin \beta x$, for every couple of coniugate complex eigenvalues $\lambda_\pm = \alpha \pm i\beta$, considered only once. Then, by imposing the differential equalities (1) and the initial conditions, one univocally gets the coefficients of the n linear combinations.

1.1 First example

Let us consider the linear dynamical system

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{pmatrix} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{pmatrix}. \tag{3}$$

The *characteristic polynomial* $\mathcal{P}(\lambda)$ of the constant matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 2 & 2 \end{pmatrix}$$

is $\mathcal{P}(\lambda) = (\lambda - 1)^2(6 - \lambda)$. From the eigenvalue $\lambda = 1$, with algebraic multiplicity 2, we obtain the bidimensional eigenspace spanned by the two eigenvectors $\boldsymbol{v}_1 = (1, 0, 0)$ and $\boldsymbol{v}_2 = (0, 1, -2)$; from the simple eigenvalue $\lambda = 6$ the eigenvector $\boldsymbol{v}_3 = (0, 2, 1)$ follows.

By using the first method, we write the solution in the form

$$\begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^x + C_2 \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} e^x + C_3 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} e^{6x}$$

and by imposing the initial conditions one gets the solution

$$\begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{pmatrix} = \begin{pmatrix} \bar{y}_1 \\ (\bar{y}_2 - 2\bar{y}_3)/5 \\ (-2\bar{y}_2 + 4\bar{y}_3)/5 \end{pmatrix} e^x + \begin{pmatrix} 0 \\ (4\bar{y}_2 + 2\bar{y}_3)/5 \\ (2\bar{y}_2 + \bar{y}_3)/5 \end{pmatrix} e^{6x} \tag{4}$$

By using the second method, we write the solution in the form

$$\begin{cases} y_1(x) = ae^x + be^6x \\ y_2(x) = ce^x + de^6x \\ y_3(x) = he^x + ke^6x \end{cases} \qquad \text{or in the form} \qquad \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{pmatrix} = \begin{pmatrix} a \\ c \\ h \end{pmatrix} e^x + \begin{pmatrix} b \\ d \\ k \end{pmatrix} e^{6x}$$

and by imposing the differential equalities (3), with the initial conditions, we obviously obtain the same solution (4).

1.2 Second example

Let us consider the linear dynamical system

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 5 & 9 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}. \tag{5}$$

The *characteristic polynomial* $\mathcal{P}(\lambda)$ of the constant matrix

$$A = \begin{pmatrix} 5 & 9 \\ -2 & -1 \end{pmatrix}$$

is $\mathcal{P}(\lambda) = \lambda^2 - 4\lambda + 13$, from which the eigenvalues $\lambda = 2 + 3i$ and $\lambda = 2 - 3i$ follow.

By using the second method, we write the solution in the form

$$\begin{cases} y_1(x) = ae^{2x} \cos 3x + be^{2x} \sin 3x \\ y_2(x) = ce^{2x} \cos 3x + de^{2x} \sin 3x \end{cases}$$

and by imposing the initial conditions, we obtain $a = \bar{y}_1$ and $c = \bar{y}_2$. If we then impose the differential equalities $\dot{y}_1 = 5y_1 + 9y_2$ and $\dot{y}_2 = -2y_1 - y_2$ on the solution

$$\begin{cases} y_1(x) = \bar{y}_1\, e^{2x} \cos 3x + be^{2x} \sin 3x \\ y_2(x) = \bar{y}_2\, e^{2x} \cos 3x + de^{2x} \sin 3x, \end{cases}$$

we obtain the solution of the dynamical system (5)

$$\begin{cases} y_1(x) = \bar{y}_1\, e^{2x} \cos 3x + (\bar{y}_1 + 3\bar{y}_2)\, e^{2x} \sin 3x \\ \\ y_2(x) = \bar{y}_2\, e^{2x} \cos 3x - \left(\dfrac{2\bar{y}_1}{3} + \bar{y}_2\right) e^{2x} \sin 3x. \end{cases}$$

2 Case of non diagonalizable matrix

If the constant matrix A is non diagonalizable, then it follows that at least an eigenvalue of the matrix has algebraic multiplicity greater than 1 with dimension of the corresponding eigenspace less than the algebraic multiplicity. For the solution we write every $y_i(x)$ as sum of two linear combinations: the former is connected to those eigenvalues having algebraic multiplicity equal to the dimension of the corresponding eigenspace and for these ones we write the same linear combinations as before; the latter is related to those eigenvalues μ having algebraic multiplicity k greater than the dimension of the corresponding eigenspace and for these ones we write the linear combination

$$a_0 e^{\mu x} + a_1 x e^{\mu x} + a_2 x^2 e^{\mu x} + \cdots + a_{k-1} x^{k-1} e^{\mu x}.$$

By imposing the differential equalities and the initial conditions, one gets finally the solution. As an example let us consider the linear dynamical system

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 5 & -9 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}. \tag{6}$$

The *characteristic polynomial* $\mathcal{P}(\lambda)$ of the constant matrix

$$A = \begin{pmatrix} 5 & -9 \\ 1 & -1 \end{pmatrix}$$

is $\mathcal{P}(\lambda) = (\lambda - 2)^2$, from which the eigenvalues $\lambda = 2$ with algebraic multiplicity 2 follows.

Since the eigenspace associated to this eigenvalue has dimension 1, the matrix A is non diagonalizable and we then write the solution in the form

$$\begin{cases} y_1(x) = ae^{2x} + bxe^{2x} \\ y_2(x) = ce^{2x} + dxe^{2x}. \end{cases}$$

From the initial conditions we get $a = \bar{y}_1$ and $c = \bar{y}_2$ and if we thereafter impose the differential equalities $\dot{y}_1 = 5y_1 - 9y_2$ and $\dot{y}_2 = y_1 - y_2$ on the solution

$$\begin{cases} y_1(x) = \bar{y}_1\, e^{2x} + bxe^{2x} \\ y_2(x) = \bar{y}_2\, e^{2x} + dxe^{2x}, \end{cases}$$

we finally obtain the solution of the dynamical system (6)

$$\begin{cases} y_1(x) = \bar{y}_1\, e^{2x} + (3\bar{y}_1 - 9\bar{y}_2)\, xe^{2x} \\ y_2(x) = \bar{y}_2\, e^{2x} + (\bar{y}_1 - 3\bar{y}_2)\, xe^{2x}. \end{cases}$$