Short overview of Financial Mathematics

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1 Exercise on the annuities

Find the *interest rate* i of the unitary period for the investment financial operation described by the following cash-flow

0	1	2	3	4	5
-1000	-2000	-2000	-2000	-2000	10000

where the negative amounts of money are paid at the periods $0, 1, 2, 3, 4$, whereas the positive one is received at time 5.

Solution

The equation of the financial operation is

$$1000r^5 + 2000r^4 + 2000r^3 + 2000r^2 + 2000r = 10000,$$

whose simplified form can be obtained after division by 1000, that is

$$r^5 + 2r^4 + 2r^3 + 2r^2 + 2r = 10, \tag{1}$$

where $r = 1 + i$. In order to solve the equation (1) numerically, we write it in two forms. We can obtain the first form by using the sum

$$2r^4 + 2r^3 + 2r^2 + 2r = 2(r^4 + r^3 + r^2 + r) = \frac{2r(r^4 - 1)}{r - 1}$$

and inserting it into the equation (1), from which the equation

$$r^5 + \frac{2r(r^4 - 1)}{r - 1} = 10$$

follows, and then, if we multiply both sides by $r-1$, the equation

$$r^5(r-1) + 2r(r^4-1) = 10(r-1),$$

that is we get the first form of the the equation (1) given by

$$r^6 + r^5 = 12r - 10. \tag{2a}$$

The second form of the equation (1) is simply

$$r^5 + 2r^4 + 2r^3 = -2r^2 - 2r + 10. \tag{2b}$$

Solution from the first form

If we now consider the first form (2a), we put $r \equiv x$, denoting the left-hand side by $f(x)$ and the right-hand side by $g(x)$, that is

$$\begin{cases} f(x) = x^6 + x^5 \\ g(x) = 12x - 10, \end{cases}$$

where $g(x)$ is the blue line (polynomial of first degree) in the figures below, while $f(x)$ is the red curve in the figures below passing through the origin and having increasing and convex behaviour because both derivatives

$$f'(x) = 6x^5 + 5x^4, \qquad f''(x) = 30x^4 + 20x^3$$

are nonnegative for nonnegative x.

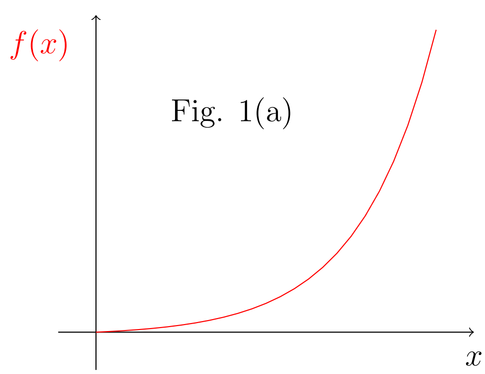


Fig. 1(a)

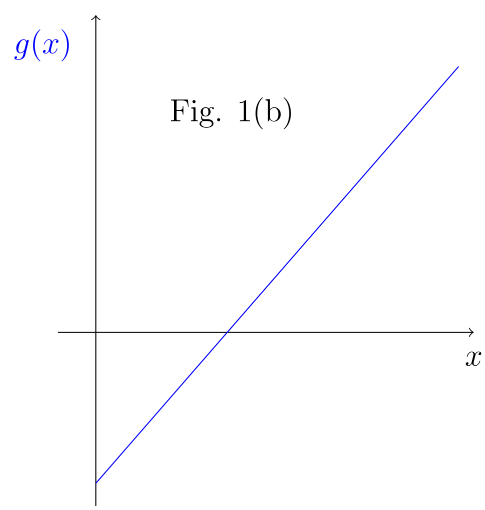


Fig. 1(b)

If we draw $f(x), g(x)$ on the same cartesian plane, we obtain from the graphic point of view that the two analytic solutions of the equation (2a) correspond to the x-coordinate of the two intersection points P, Q of the two curves $f(x), g(x)$.

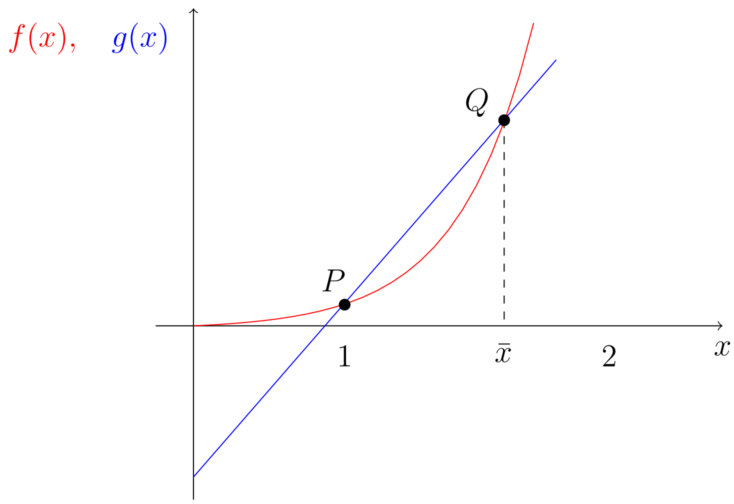


Fig. 1(c)

We have that the intersection point P has x-coordinate $x = 1$ because it yields $f(1) = g(1) = 2$ and the inequality $f'(1) = 11 < 12 = g'(1)$ holds, that is the slope of the tangent line to the curve $f(x)$ in $x = 1$ is less than the slope of the line $g(x)$. From the comparison between the slopes of the tangent line of the curve $f(x)$ in $x = 1$ and the slope of the line $g(x)$, we get that there exists a second intersection point Q whose x-coordinate is less than 2 because for $x = 2$ the value $f(2) = 96$ on the red curve is greater than the value $g(2) = 14$ on the blue curve.

The solution with financial consistency is the x-coordinate $\bar{x}$ of the intersection point Q in fig. 1(c) because it yields $\bar{x} > 1$, from which we get the *interest rate* $i = \bar{x} - 1$. In order to find the approximated value of $\bar{x}$ with Excel, we use the following algorithm by considering fig. 1(c):

- we choose a value x_1 belonging to the interval $(1, 2)$;
- we insert the value x_1 into both functions $f(x), g(x)$;
- if $f(x_1) < g(x_1)$, we choose the next value x_2 such that $x_2 > x_1$ and we repeat the operations with the value x_2;
- if $f(x_1) > g(x_1)$, we choose the next value x_2 such that $x_2 < x_1$ and we repeat the operations with the value x_2.

Solution from the second form

If we now consider the second form (2b), we put $r \equiv x$, always denoting the left-hand side by $f(x)$ and the right-hand side by $g(x)$, that is

$$\begin{cases} f(x) = x^5 + 2x^4 + 2x^3 \\ g(x) = -2x^2 - 2x + 10, \end{cases}$$

where $g(x)$ is the blue parabola (polynomial of second degree) in the figures below, while $f(x)$ is the red curve in the figures below passing through the origin and having increasing and convex behaviour because both derivatives

$$f'(x) = 5x^4 + 8x^3 + 6x^2, \qquad f''(x) = 20x^3 + 24x^2 + 12x$$

are nonnegative for nonnegative x.

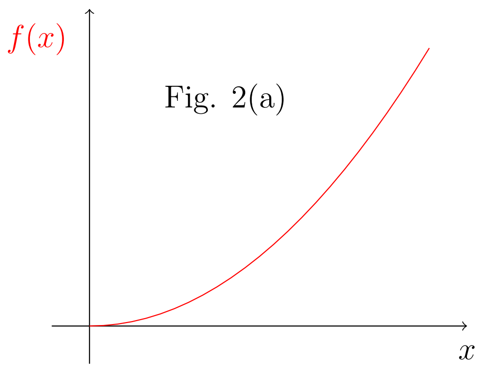


Fig. 2(a)

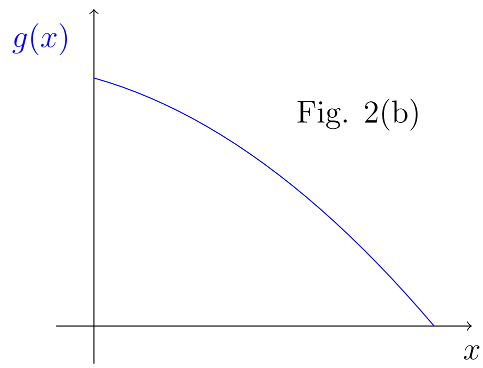


Fig. 2(b)

If we draw $f(x), g(x)$ on the same cartesian plane, we obtain from the graphic point of view that the analytic solution of the equation (2b) corresponds to the x-coordinate, denoted by $\bar{x}$, of the intersection point P of the two curves $f(x), g(x)$.

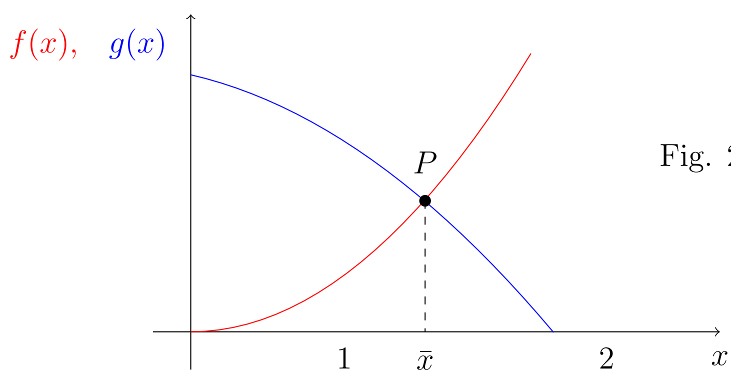


Fig. 2(c)

We observe that the inequality $1 < \bar{x} < 2$ holds because it yields

$$f(1) = 5 < 6 = g(1) \qquad \text{and} \qquad f(2) = 80 > -2 = g(2).$$

In order to find the approximated value of $\bar{x}$ with Excel, we use the following algorithm by considering fig. 2(c):

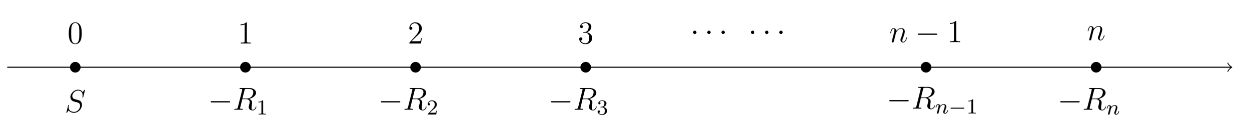
- we choose a value x_1 belonging to the interval $(1, 2)$;
- we insert the value x_1 into both functions $f(x), g(x)$;
- if $f(x_1) < g(x_1)$, we choose the next value x_2 such that $x_2 > x_1$ and we repeat the operations with the value x_2;
- if $f(x_1) > g(x_1)$, we choose the next value x_2 such that $x_2 < x_1$ and we repeat the operations with the value x_2.

2 General theory of the amortization of a loan

The amortization of a borrowed *principal (loan)*, denoted by S and taken from a bank at time $t = 0$, consists of determining at every fixed time k the following four quantities:

- the *payment amount*, denoted by R_k,
- the *remaining principal*, or *remaining debt*, denoted by D_k,
- the *principal paid*, denoted by C_k,
- the *interest paid*, denoted by I_k.

If we consider a borrowed principal S, taken from a bank a time $t = 0$, and a sequence of payment amounts R_k which must be paid every unit period $t = 1, 2, 3, \ldots, n-1, n$, we can represent the financial operation through the following cash-flow.



If we put $v = 1/(1 + i)$, the financial equivalence is given by the equality

$$S = R_1 v + R_2 v^2 + R_3 v^3 + \cdots + R_{n-1} v^{n-1} + R_n v^n = \sum_{h=1}^{n} R_h v^h. \tag{3}$$

For every time $t \equiv k = 1, 2, 3, \ldots, n-1, n$, we now define the *remaining principal*, or *remaining debt*, denoted by D_k, of the amortization as

$$D_k = \frac{1}{v^k} \left(S - \sum_{h=1}^{k} R_h v^h \right), \tag{4}$$

that, by substitution of (3), assumes the form

$$D_k = \frac{1}{v^k}\left(S - \sum_{h=1}^{k} R_h v^h\right) = \frac{1}{v^k}\left(\sum_{h=1}^{n} R_h v^h - \sum_{h=1}^{k} R_h v^h\right) = \sum_{h=k+1}^{n} R_h v^{h-k},$$

from which we get the *remaining debt* D_{k-1} at time $t = k-1$

$$D_{k-1} = \sum_{h=k}^{n} R_h v^{h-k+1}. \tag{5}$$

It is straightforward to notice that the *remaining debt* D_k, defined by (4) where S is given by (3), is obviously decreasing and satisfies both accounting properties $D_0 = S$ and $D_n = 0$.

The significance and the interpretation of the definition (4) are straightforward: the quantity D_k given by (4) effectively represents the *remaining debt* at time k because it is the difference between the initial debt S and what one has already paid until time k, calculated at time k through the multiplication by the accumulation factor $1/v^k \equiv (1+i)^k$.

Further, for every time $t \equiv k = 1, 2, 3, \ldots, n-1, n$, we define the *principal paid*, denoted by C_k, of the amortization through the relation

$$\sum_{h=1}^{k} C_h = S - D_k\,, \tag{6}$$

from which we get the accounting property corresponding to $D_n = 0$

$$\sum_{h=1}^{n} C_h = S - D_n = S, \tag{7}$$

the *remaining debt*

$$D_k = S - \sum_{h=1}^{k} C_h \tag{8}$$

and then the difference between two consecutive *remaining debts*

$$D_{k-1} - D_k = \left(S - \sum_{h=1}^{k-1} C_h\right) - \left(S - \sum_{h=1}^{k} C_h\right) = \sum_{h=1}^{k} C_h - \sum_{h=1}^{k-1} C_h = C_k\,. \tag{9}$$

Finally, for every time $t \equiv k = 1, 2, 3, \ldots, n-1, n$, we define the *interest paid*, denoted by I_k, as the difference

$$I_k = R_k - C_k\,, \tag{10}$$

which satisfies $I_k = iD_{k-1}$ because, by virtue of (5) and of $(1-v)/v = i$, it yields

$$I_k = R_k - C_k = R_k - D_{k-1} + D_k = R_k - \sum_{h=k}^{n} R_h v^{h-k+1} + \sum_{h=k+1}^{n} R_h v^{h-k} =$$

$$= R_k - R_k v - v \sum_{h=k+1}^{n} R_h v^{h-k} + \sum_{h=k+1}^{n} R_h v^{h-k} = R_k(1-v) + \frac{1-v}{v^k} \sum_{h=k+1}^{n} R_h v^h =$$

$$= \frac{1-v}{v^k} \left(R_k v^k + \sum_{h=k+1}^{n} R_h v^h \right) = \frac{1-v}{v^k} \sum_{h=k}^{n} R_h v^h = \frac{1-v}{v} \sum_{h=k}^{n} R_h v^{h-k+1} = iD_{k-1}\,,$$

that is

$$I_k = iD_{k-1}\,. \tag{11}$$

In the case the problem data are the initial loan S and a condition about the *payment amounts* R_k, then we perform the following steps according the following sequence:

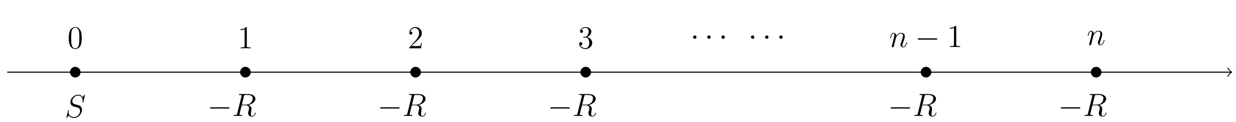
- we determine the *payment amounts* R_k from equation (3);
- we determine the *remaining debt* D_k from equation (4);
- we determine the *principal paid* C_k from equation (6);
- we determine the *interest paid* I_k from equation (10).

In the case the problem data are the initial loan S and a condition about the *principal paid* C_k, then we perform the following steps according the following sequence:

- we determine the *principal paid* C_k from the given conditions on the *principal paid*;
- we determine the *remaining debt* D_k from equation (8);
- we determine the *payment amounts* R_k from equation (4);
- we determine the *interest paid* I_k from equation (10).

2.1 French amortization

The french amortization of a borrowed *principal (loan)* is characterized by a constant *payment amount.* If we consider a borrowed principal S, taken from a bank a time $t = 0$, and a constant payment amount R which must be paid every unit period $t = 1, 2, 3, \ldots, n-1, n$, we can represent the financial operation through the following cash-flow.



If we put $v = 1/(1+i)$, the financial equivalence (3) assumes the form

$$S = Rv + Rv^2 + Rv^3 + \cdots + Rv^{n-1} + Rv^n,$$

that is

$$S = \frac{v(1-v^n)}{1-v}\, R,$$

from which the value of payment amount

$$R = \frac{1-v}{v(1-v^n)}\, S \tag{12}$$

follows, where we have used the formula of the sum

$$1 + v + v^2 + v^3 + \cdots + v^{n-2} + v^{n-1} = \frac{1-v^n}{1-v}\,. \tag{13}$$

For every time $t \equiv k = 1, 2, 3, \ldots, n-1, n$, the *remaining principal*, or *remaining debt*, denoted by D_k, of the amortization, defined by (4), takes the form

$$D_k = \frac{1}{v^k}\left(S - \sum_{h=1}^{k} Rv^h\right), \tag{14a}$$

whose expansion is

$$D_k = \frac{1}{v^k}\left(S - \sum_{h=1}^{k} Rv^h\right) = \frac{S - Rv - Rv^2 - Rv^3 - \cdots - Rv^k}{v^k} =$$

$$= \frac{S - Rv\left(1 + v + v^2 + v^3 + \cdots + v^{k-2} + v^{k-1}\right)}{v^k} =$$

$$= \frac{S}{v^k} - \frac{Rv\left(1-v^k\right)}{v^k(1-v)} = \frac{S}{v^k} - \left[\frac{1-v}{v(1-v^n)}\, S\right] \frac{v\left(1-v^k\right)}{v^k(1-v)} = \frac{1-v^{n-k}}{1-v^n}\, S,$$

that is

$$D_k = \frac{1-v^{n-k}}{1-v^n}\, S, \tag{14b}$$

which satisfies both accounting properties $D_0 = S$ and $D_n = 0$.

Further, for every time $t \equiv k = 1, 2, 3, \ldots, n-1, n$, the *principal paid*, denoted by C_k, of the amortization, defined by (6), allows to compute the expressions C_i, inductively.

For $k = 1$ we obtain

$$C_1 = S - D_1 = S - \frac{1-v^{n-1}}{1-v^n}\, S = \frac{v^{n-1} - v^n}{1-v^n}\, S,$$

that is

$$C_1 = \frac{v^{n-1} - v^n}{1-v^n}\, S; \tag{15a}$$

for $k=2$ we have $C_1+C_2=S-D_2$, that is

$$C_2 = S - C_1 - D_2 = S - \frac{v^{n-1}-v^n}{1-v^n}\,S - \frac{1-v^{n-2}}{1-v^n}\,S =$$

$$= \left[\frac{1-v^n-v^{n-1}+v^n-1+v^{n-2}}{1-v^n}\right]S = \frac{v^{n-2}-v^{n-1}}{1-v^n}\,S =$$

$$= \frac{v^{n-1}-v^n}{v(1-v^n)}\,S = \frac{C_1}{v} = C_1(1+i);$$

for $k=3$ we have $C_1+C_2+C_3=S-D_3$, that is

$$C_3 = S - C_1 - C_2 - D_3 = S - \frac{v^{n-1}-v^n}{1-v^n}\,S - \frac{v^{n-1}-v^n}{v(1-v^n)}\,S - \frac{1-v^{n-3}}{1-v^n}\,S =$$

$$= \left[\frac{v-v^{n+1}-v^n+v^{n+1}-v^{n-1}+v^n-v+v^{n-2}}{v(1-v^n)}\right]S =$$

$$= \frac{v^{n-2}-v^{n-1}}{v(1-v^n)}\,S = \frac{v^{n-1}-v^n}{v^2(1-v^n)}\,S = \frac{C_1}{v^2} = C_1(1+i)^2,$$

from which, by induction, we get the expression of the *principal paid*

$$C_k = \frac{v^{n-1}-v^n}{v^{k-1}(1-v^n)}\,S = \frac{C_1}{v^{k-1}} = C_1(1+i)^{k-1}, \tag{15b}$$

which satisfies both accounting properties (7) and (9) because it yields

$$\sum_{k=1}^{n} C_k = C_1+C_2+C_3+\cdots+C_{n-1}+C_n = C_1 + \frac{C_1}{v} + \frac{C_1}{v^2} + \frac{C_1}{v^3} + \cdots + \frac{C_1}{v^{n-2}} + \frac{C_1}{v^{n-1}} =$$

$$= \frac{C_1}{v^{n-1}}\left(1+v+v^2+v^3+\cdots+v^{n-2}+v^{n-1}\right) = \frac{C_1}{v^{n-1}}\,\frac{1-v^n}{1-v} = \frac{v^{n-1}-v^n}{v^{n-1}(1-v^n)}\,S\,\frac{1-v^n}{1-v} = S,$$

that is

$$\sum_{k=1}^{n} C_k = S,$$

and

$$D_{k-1}-D_k = \frac{1-v^{n-k+1}}{1-v^n}\,S - \frac{1-v^{n-k}}{1-v^n}\,S = \frac{v^{n-k}-v^{n-k+1}}{1-v^n}\,S = \frac{v^{n-1}-v^n}{v^{k-1}(1-v^n)}\,S = \frac{C_1}{v^{k-1}} = C_k\,,$$

that is $D_{k-1}-D_k=C_k$. The expression (15b) shows that the *principal paid* C_k of the *french amortization* increases from C_1 in *geometric progression*.

Finally, for every time $t \equiv k = 1,2,3,\ldots,n-1,n$, the *interest paid*, denoted by I_k, defined by (10), is given by

$$I_k = R - C_k\,, \tag{16}$$

whose expansion is

$$I_k = R - C_k = \frac{1-v}{v(1-v^n)}\,S - \frac{v^{n-1}-v^n}{v^{k-1}(1-v^n)}\,S = \frac{v^{k-2}-v^{k-1}-v^{n-1}+v^n}{v^{k-1}(1-v^n)}\,S,$$

and it is straightforward to notice that the relation $I_k = iD_{k-1}$ in (11) holds because we have

$$iD_{k-1} = \left(\frac{1}{v}-1\right)D_{k-1} = \left(\frac{1}{v}-1\right)\frac{1-v^{n-k+1}}{1-v^n}\,S = \frac{(1-v)(1-v^{n-k+1})}{v(1-v^n)}\,S =$$

$$= \frac{v^{k-2}(1-v)(1-v^{n-k+1})}{v^{k-1}(1-v^n)}\,S = \frac{v^{k-2}-v^{k-1}-v^{n-1}+v^n}{v^{k-1}(1-v^n)}\,S = I_k\,.$$

2.2 Italian amortization

The italian amortization of a loan S is characterized by the costant *principal paid*, denoted by C, from which, by virtue of the property (7), we get $C = S/n$.

From the equation (6) defining the *principal paid* C_k, we get

$$D_k = S - \sum_{h=1}^{k} C_k = S - \sum_{h=1}^{k} C = S - \sum_{h=1}^{k}\frac{S}{n} = S - \frac{kS}{n} = \frac{n-k}{n}\,S,$$

that is

$$D_k = \frac{n-k}{n}\,S. \tag{17}$$

From the equation (4), rewritten in the form

$$\sum_{h=1}^{k} R_h v^h = S - v^k D_k$$

and connecting the *remaining debt* D_k and the *payment amount* R_k, we can obtain R_k for every time $t \equiv k = 1, 2, 3, \ldots, n-1, n$, inductively. For $k = 1$ we have

$$R_1 = \frac{S - vD_1}{v} = \frac{S}{v} - D_1 = \frac{S}{v} - \frac{n-1}{n}\,S = \left(\frac{1-v}{v} + \frac{1}{n}\right)S = \left(\frac{1}{v} - 1 + \frac{1}{n}\right)S;$$

for $k = 2$ we have

$$R_2 = \frac{S - v^2 D_2 - R_1 v}{v^2} = \frac{S}{v^2} - D_2 - \frac{R_1}{v} = \frac{S}{v^2} - \frac{n-2}{n}\,S - \left(\frac{1}{v} - 1 + \frac{1}{n}\right)\frac{S}{v} =$$

$$= \left(\frac{1}{v^2} - 1 + \frac{2}{n} - \frac{1}{v^2} + \frac{1}{v} - \frac{1}{nv}\right)S = \left(\frac{1}{v} - 1 + \frac{1}{n}\right)S - \left(\frac{1}{nv} - \frac{1}{n}\right)S = R_1 - \left(\frac{1}{nv} - \frac{1}{n}\right)S;$$

for $k = 3$ we have

$$R_3 = \frac{S - v^3 D_3 - R_1 v - R_2 v^2}{v^3} = \frac{S}{v^3} - D_3 - \frac{R_1}{v^2} - \frac{R_2}{v} =$$

$$= \left(\cancel{\frac{1}{v^3}} - \frac{n-3}{n} - \cancel{\frac{1}{v^3}} + \cancel{\frac{1}{v^2}} - \cancel{\frac{1}{nv^2}} - \cancel{\frac{1}{v^2}} + \frac{1}{v} - \frac{1}{nv} + \cancel{\frac{1}{nv^2}} - \frac{1}{nv} \right) S =$$

$$= \left(-\frac{n-3}{n} + \frac{1}{v} - \frac{2}{nv} \right) S = \left(\frac{1}{v} - 1 + \frac{1}{n} - \frac{2}{nv} + \frac{2}{n} \right) S =$$

$$= \left(\frac{1}{v} - 1 + \frac{1}{n} \right) S - 2 \left(\frac{1}{nv} - \frac{1}{n} \right) S = R_1 - 2 \left(\frac{1}{nv} - \frac{1}{n} \right) S,$$

from which we inductively get the expressione at any time $t = k$

$$R_k = R_1 - (k-1) \left(\frac{1}{nv} - \frac{1}{n} \right) S. \tag{18}$$

The expression (18) shows that the *payment amount* R_k of the *italian amortization* decreases from R_1 in *arithmetic progression*. We can verify that the *payment amount* R_k of the *italian amortization* satisfies the financial equivalence condition

$$\sum_{k=1}^{n} R_k v^k = S, \tag{19}$$

for which we need the formula

$$\sum_{k=1}^{n} k v^k = v \sum_{k=1}^{n} k v^{k-1} = v \sum_{k=1}^{n} \frac{d}{dv} v^k = v \frac{d}{dv} \sum_{k=1}^{n} v^k = v \frac{d}{dv} \left(v \frac{1 - v^n}{1 - v} \right) =$$

$$= v \left[\frac{1 - v^n}{1 - v} + v \frac{-n v^{n-1}(1 - v) + 1 - v^n}{(1 - v)^2} \right] = \frac{v - v^{n+1} - n v^{n+1} + n v^{n+2}}{(1 - v)^2},$$

that is

$$\sum_{k=1}^{n} k v^k = \frac{v - v^{n+1} - n v^{n+1} + n v^{n+2}}{(1 - v)^2}. \tag{20}$$

By applying the formula (20) and the formula (13), we have

$$\sum_{k=1}^{n} R_k v^k = \left[R_1 + \left(\frac{1}{nv} - \frac{1}{n} \right) S \right] \sum_{k=1}^{n} v^k - \left(\frac{1}{nv} - \frac{1}{n} \right) S \sum_{k=1}^{n} k v^k =$$

$$= S \left[\left(\frac{1}{v} - 1 + \cancel{\frac{1}{n}} + \frac{1}{nv} - \cancel{\frac{1}{n}} \right) \sum_{k=1}^{n} v^k - \left(\frac{1}{nv} - \frac{1}{n} \right) \sum_{k=1}^{n} k v^k \right] =$$

$$= S \left[\left(1 - v + \frac{1}{n} \right) \frac{1 - v^n}{1 - v} - \left(\frac{1}{nv} - \frac{1}{n} \right) \frac{v - v^{n+1} - n v^{n+1} + n v^{n+2}}{(1 - v)^2} \right] =$$

$$= S\left[\left(\frac{n-nv+1}{n}\right)\frac{1-v^n}{1-v} - \left(\frac{1-v}{n}\right)\frac{1-v^n-nv^n+nv^{n+1}}{(1-v)^2}\right] =$$

$$= S\left[\left(\frac{n-nv+1}{n}\right)\frac{1-v^n}{1-v} - \frac{1-v^n-nv^n+nv^{n+1}}{n(1-v)}\right] =$$

$$= \frac{(n-nv+1)(1-v^n)-1+v^n+nv^n-nv^{n+1}}{n(1-v)}\,S =$$

$$= \frac{n-\cancel{nv^n}-nv+\cancel{nv^{n+1}}+\cancel{1}-\cancel{v^n}-\cancel{1}+\cancel{v^n}+\cancel{nv^n}-\cancel{nv^{n+1}}}{n(1-v)}\,S = \frac{n(1-v)}{n(1-v)}\,S = S,$$

that is we have proved the equality (19).