MATHEMATICS FOR FINANCE Exam

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Surname ____________________________ **Name** ____________________________

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Exercise 1. Given the canonical basis $\mathcal{B}_{\mathbb{R}^3} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ of the vector space $\mathbb{R}^3$ and the linear application $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ acting on the basis vectors of $\mathbb{R}^3$ according the transformation laws

$$\begin{cases} L(\boldsymbol{e}_1) = -14\boldsymbol{e}_1 - 10\boldsymbol{e}_2 + 6\boldsymbol{e}_3 \\ L(\boldsymbol{e}_2) = -6\boldsymbol{e}_1 - 4\boldsymbol{e}_2 + 2\boldsymbol{e}_3 \\ L(\boldsymbol{e}_3) = 2\boldsymbol{e}_1 + 2\boldsymbol{e}_2 - 2\boldsymbol{e}_3 \end{cases}$$

1) write the matrix A associated to the linear application L with respect to the given basis;

2) find the subspaces *kernel* and *image* of the linear application L determining a basis for both subspaces.

Let us consider the linear application $\tilde{L} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by the transformation laws of the components

$$\tilde{L}(x, y, z) = (y, -y + z, x + z),$$

where in the vector space $\mathbb{R}^3$ the same basis $\mathcal{B}_{\mathbb{R}^3}$ as before is fixed.

3) Write the matrix B associated to the linear application $\tilde{L}$ with respect to the given basis and determine the matrix, denoted by M, associated to the product of linear applications in the order $L\tilde{L}$.

4) Verify whether the matrix M is diagonalizable.

If M is diagonalizable,

5) find the basis vectors with respect to which the matrix M assumes a diagonal form denoted by $\mathcal{D}$ and write the matrix C of the basis change such that $C^{-1}MC = \mathcal{D}$;

6) write the diagonal matrix $\mathcal{D}$ (without performing the matrix multiplication $C^{-1}MC$);

7) in the eigenspace corresponding to the eigenvalue of algebraic multiplicity 2, find an eingenvector orthogonal to the vector $\boldsymbol{v} = (1, 4, -2)$.

Exercise 2. Solve the following Cauchy problem

$$\begin{cases} y''(x) - 2y'(x) + y(x) = 2e^x - 6xe^x \\ y(0) = -1 \\ y'(0) = 2 \end{cases}$$

Exercise 3. Find the optimal points of the function

$$f(x, y, z) = \frac{1}{3}x + \frac{1}{2}y + \frac{1}{2}z$$

subject to the constraints $2xy = 1$ and $3yz = 2$.

Solution

Exercise 1.

1) The matrix A is

$$A = \begin{pmatrix} -14 & -6 & 2 \\ -10 & -4 & 2 \\ 6 & 2 & -2 \end{pmatrix},$$

obtained by writing in columns the coefficients of

$$-14\boldsymbol{e}_1 - 10\boldsymbol{e}_2 + 6\boldsymbol{e}_3, \qquad -6\boldsymbol{e}_1 - 4\boldsymbol{e}_2 + 2\boldsymbol{e}_3, \qquad 2\boldsymbol{e}_1 + 2\boldsymbol{e}_2 - 2\boldsymbol{e}_3.$$

2) The *kernel* of L is the subspace of $\mathbb{R}^3$ containing the vectors $\boldsymbol{v} = (x, y, z)$ such that it yields $L(\boldsymbol{v}) = \mathbf{0}$, that is

$$\begin{pmatrix} -14 & -6 & 2 \\ -10 & -4 & 2 \\ 6 & 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which is an algebraic linear system having *rank* 2, because

$$\det \begin{pmatrix} -14 & -6 & 2 \\ -10 & -4 & 2 \\ 6 & 2 & -2 \end{pmatrix} = 0,$$

and the *minor* of order 2

$$\begin{pmatrix} -4 & 2 \\ 2 & -2 \end{pmatrix},$$

highlighted in the matrix A

$$A = \begin{pmatrix} -14 & -6 & 2 \\ -10 & \boxed{-4} & \boxed{2} \\ 6 & \boxed{2} & \boxed{-2} \end{pmatrix},$$

has determinant not equal to zero. From this *minor* we get the system

$$\begin{cases} -4y + 2z = 10t \\ 2y - 2z = -6t, \end{cases}$$

in which we have given the arbitrary value $x = t$ to the unknown x, that lays outside the *minor* highlighted in the matrix A. The *kernel* has then dimension 1 because this linear system has ∞^1 solutions which are

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} t,$$

from which we get that the *kernel* has basis vector $(1, -2, 1)$. The *image* of L is spanned by all those column vectors having some component contained inside the *minor* highlighted in the matrix A, that is we have the basis of the *image*

$$\mathcal{B}_{Im(L)} = \left\{ \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\},$$

where the second and third column of A have been divided by 2.

3) The matrix B associated to the linear application $\tilde{L}(x, y, z) = (y, -y + z, x + z)$ is the matrix

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

because it reproduces the transformation laws of $\tilde{\mathcal{L}}$, that is

$$\tilde{\mathcal{L}}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ -y+z \\ x+y \end{pmatrix}.$$

From the matrix B, one gets the matrix M associated to the product of linear applications in the order $L\tilde{L}$

$$M = AB = \begin{pmatrix} -14 & -6 & 2 \\ -10 & -4 & 2 \\ 6 & 2 & -2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -8 & -4 \\ 2 & -6 & -2 \\ -2 & 4 & 0 \end{pmatrix}.$$

4,5) In order to verify whether the matrix M, which is an *endomorphism* of $\mathbb{R}^3$, is *diagonalizable*, we have to extablish whether there exists a basis of the vector space $\mathbb{R}^3$ consisting of three eigenvectors of M, that is we have to verify, in other words, whether there exist three *linearly independent* eigenvectors of M.

The *characteristic polynomial* of M is

$$\det\begin{pmatrix} 2-\lambda & -8 & -4 \\ 2 & -6-\lambda & -2 \\ -2 & 4 & -\lambda \end{pmatrix} = -\lambda(\lambda^2+4\lambda+4),$$

whose zeros are the *simple*[1] eigenvalue $\lambda = 0$ and the eigenvalue $\lambda = -2$, having *algebraic multiplicity* 2.

To the *simple* eigenvalue $\lambda = 0$, we associate the linear system $(M - 0\mathbb{I})\boldsymbol{u} = \mathbf{0}$, that is

$$\begin{pmatrix} 2 & -8 & -4 \\ 2 & -6 & -2 \\ -2 & 4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

whose ∞^1 solutions, by virtue of the *minor* of order 2

$$\begin{pmatrix} 2 & -8 & -4 \\ 2 & \boxed{\begin{matrix} -6 & -2 \\ 4 & 0 \end{matrix}} \\ -2 & & \end{pmatrix},$$

are $x = t, y = t, z = t$, from which we get the eigenvector $\boldsymbol{u} = (2, 1, -1)$, satisfying effectively the equality

$$\begin{pmatrix} 2 & -8 & -4 \\ 2 & -6 & -2 \\ -2 & 4 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 0\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \qquad \text{that is} \qquad M\boldsymbol{u} = 0\boldsymbol{u}.$$

To the eigenvalue $\lambda = -2$, we associate the linear system $[M - (-2)\mathbb{I}]\boldsymbol{w} = \mathbf{0}$, that is

$$\begin{pmatrix} 4 & -8 & -4 \\ 2 & -4 & -2 \\ -2 & 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

whose ∞^2 solutions, by virtue of the *minor* of order 1

$$\begin{pmatrix} \boxed{4} & -8 & -4 \\ 2 & -4 & -2 \\ -2 & 4 & 2 \end{pmatrix}$$

[1]An eigenvalue λ of a matrix is called *simple eigenvalue* if its *algebraic multiplicity* is 1.

are

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \alpha + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \beta, \tag{1}$$

from which we get the two eigenvectors $\boldsymbol{w}_1 = (2, 1, 0)$ and $\boldsymbol{w}_2 = (1, 0, 1)$, satisfying the equalities

$$\begin{pmatrix} 2 & -8 & -4 \\ 2 & -6 & -2 \\ -2 & 4 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = -2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} 2 & -8 & -4 \\ 2 & -6 & -2 \\ -2 & 4 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

that is $M\boldsymbol{w}_1 = -2\boldsymbol{w}_1$ and $M\boldsymbol{w}_2 = -2\boldsymbol{w}_2$.

Since the set $\mathcal{B} = \{\boldsymbol{u}, \boldsymbol{w}_1, \boldsymbol{w}_2\}$, containing three eigenvectors of the matrix M, is linearly independent, we conclude that the set $\mathcal{B}$ is a basis of the vector space $\mathbb{R}^3$, and then that the matrix M is *diagonalizable*.

The matrix C of the basis change to the basis of the eigenvectors, with respect to which M assumes diagonal form, is then the one whose columns are the eigenvectors, that is

$$C = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

6) Since we have written the eigenvectors in the matrix C in the sequence corresponding to the eigenvalues in the order $\lambda = 0, -2, -2$, respectively, it follows that the diagonal matrix $\mathcal{D}$, associated to M, is

$$\mathcal{D} = C^{-1}MC = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

7) The eigenspace associated to the eigenvalue of algebraic multiplicity 2 is the one corresponding to $\lambda = -2$, spanned by the two eigenvectors $\boldsymbol{w}_1, \boldsymbol{w}_2$, that we denote by $\mathbb{E}_{-2}$. The vectors of this subspace have the form (1), and the vector of this subspace, orthogonal to the given vector $\boldsymbol{v}$, is the vector $\boldsymbol{w} = (2\alpha + \beta, \alpha, \beta)$ such that the *scalar product* $(\boldsymbol{v}, \boldsymbol{w})$ vanishes, that is

$$(\boldsymbol{v}, \boldsymbol{w}) = (1, 4, -2) \cdot (2\alpha + \beta, \alpha, \beta) = 0,$$

from which we get the relation $6\alpha - \beta = 0$. By choosing the particular solution given by $\alpha = 1, \beta = 6$, we finally obtain the particular vector $\boldsymbol{w} = (8, 1, 6)$ belonging to $\mathbb{E}_{-2}$ and orthogonal to the given vector $\boldsymbol{v} = (1, 4, -2)$.

Exercise 2.

The homogeneous equation associated to the given equation is

$$y''(x) - 2y'(x) + y(x) = 0,$$

to which the algebraic equation

$$\lambda^2 - 2\lambda + 1 = 0$$

corresponds, having the solution $\lambda = 1$ with algebraic multiplicity 2. The solution, that we denote by $y_0(x)$, of the homogeneous equation is then

$$y_0(x) = Ae^x + Bxe^x,$$

and since the right-hand side of the given equation is $(2 - 6x)e^x$, that is the product of a polynomial of first degree times the exponential e^x, we write the *particular solution* $y_p(x)$ in the same form

$$y_p(x) = (ax + b)e^x.$$

Since this $y_p(x)$ is similar to the solution of the homogeneous equation, we multiply $y_p(x)$ times x, and we obtain the new *particular solution*

$$y_p(x) = (ax^2 + bx)e^x,$$

whose term bxe^x is similar to Bxe^x of the solution of the homogeneous equation. We have then to multiply for another factor x in such a way that the *particular solution* $y_p(x)$ assumes the final form

$$y_p(x) = (ax^3 + bx^2)e^x,$$

whose derivatives are

$$y_p'(x) = (3ax^2 + 2bx + ax^3 + bx^2)e^x \qquad \text{and} \qquad y_p''(x) = (6ax + 2b + 6ax^2 + 4bx + ax^3 + bx^2)e^x.$$

By inserting $y_p(x), y_p'(x), y_p''(x)$ into the given equation, we get the equality

$$(6ax + 2b + 6ax^2 + 4bx + ax^3 + bx^2)e^x - 2(3ax^2 + 2bx + ax^3 + bx^2)e^x + (ax^3 + bx^2)e^x = (2 - 6x)e^x,$$

that, after the semplifications

$$(6ax + 2b + \cancel{6ax^2} + 4bx + \cancel{ax^3} + \cancel{bx^2})e^x - 2(\cancel{3ax^2} + 2bx + \cancel{ax^3} + \cancel{bx^2})e^x + (\cancel{ax^3} + \cancel{bx^2})e^x = (2 - 6x)e^x,$$

becomes

$$(6ax + 2b)e^x = (2 - 6x)e^x,$$

from which we obtain the two equations $6a = -6, 2b = 2$, and then $a = -1, b = 1$.

The solution of the given differential equation is then

$$y(x) = Ae^x + Bxe^x - x^3e^x + x^2e^x,$$

whose first derivative is

$$y'(x) = Ae^x + Be^x + Bxe^x - 3x^2e^x - x^3e^x + 2xe^x + x^2e^x,$$

from which, by imposing the *initial conditions* $y(0) = -1, y'(0) = 2$ of the *Cauchy problem*, the system

$$\begin{cases} A &= -1 \\ A + B &= 2 \end{cases}$$

follows, having solution $A = -1, B = 3$. The solution of the given *Cauchy problem* is then

$$y(x) = -e^x + 3xe^x - x^3e^x + x^2e^x.$$

Exercise 3. The *Lagrangian function* $\mathcal{L}$ associated to the given optimization problem is

$$\mathcal{L}(e, y, z; \lambda, \mu) = \frac{1}{3}x + \frac{1}{2}y + \frac{1}{2}z + \lambda(2xy - 1) + \mu(3yz - 2),$$

from which the *first order conditions*

$$\begin{cases} 1/3 + 2\lambda y = 0 \\ 1/2 + 2\lambda x + 3\mu z = 0 \\ 1/2 + 3\mu y = 0 \\ 2xy = 1 \\ 3yz = 2. \end{cases}$$

From the first, third, fourth, fifth equation, we get

$$\lambda = -1/(6y), \qquad \mu = -1/(6y), \qquad x = 1/(2y), \qquad z = 2/(3y),$$

respectively, that, inserted into the second equation, give

$$\frac{1}{2} + 2\left(-\frac{1}{6y}\right)\left(\frac{1}{2y}\right) + 3\left(-\frac{1}{6y}\right)\left(\frac{2}{3y}\right) = 0 \qquad \Longrightarrow \qquad \frac{y^2-1}{2y^2} = 0,$$

where $y \neq 0$ because $y = 0$ is not consistent with the constraints. From $y^2 - 1 = 0$, we get $y = \pm 1$ and then the *optimal points*

$$A = \left(\frac{1}{2}, 1, \frac{2}{3}; -\frac{1}{6}, -\frac{1}{6}\right) \qquad \text{and} \qquad B = \left(-\frac{1}{2}, -1, -\frac{2}{3}; \frac{1}{6}, \frac{1}{6}\right).$$

The *bordered hessian matrix* is

$$\overline{H}(x,y,z;\lambda,\mu) = \begin{pmatrix} 0 & 0 & 2y & 2x & 0 \\ 0 & 0 & 0 & 3z & 3y \\ 2y & 0 & 0 & 2\lambda & 0 \\ 2x & 3z & 2\lambda & 0 & 3\mu \\ 0 & 3y & 0 & 3\mu & 0 \end{pmatrix},$$

from which we get

$$\overline{H}(A) = \begin{pmatrix} 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 3 \\ 2 & 0 & 0 & -1/3 & 0 \\ 1 & 2 & -1/3 & 0 & -1/2 \\ 0 & 3 & 0 & -1/2 & 0 \end{pmatrix} \qquad \text{and} \qquad \overline{H}(B) = \begin{pmatrix} 0 & 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & -2 & -3 \\ -2 & 0 & 0 & 1/3 & 0 \\ -1 & -2 & 1/3 & 0 & 1/2 \\ 0 & -3 & 0 & 1/2 & 0 \end{pmatrix}.$$

From

$$\det \overline{H}(A) = 2\det\begin{pmatrix} 0 & 0 & 2 & 3 \\ 2 & 0 & -1/3 & 0 \\ 1 & 2 & 0 & -1/2 \\ 0 & 3 & -1/2 & 0 \end{pmatrix} - \det\begin{pmatrix} 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & 0 \\ 1 & 2 & -1/3 & -1/2 \\ 0 & 3 & 0 & 0 \end{pmatrix} =$$

$$= 2(2)\det\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & -1/2 \\ 0 & 3 & 0 \end{pmatrix} - 2(3)\det\begin{pmatrix} 2 & 0 & -1/3 \\ 1 & 2 & 0 \\ 0 & 3 & -1/2 \end{pmatrix} + 3\det\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & -1/3 \\ 0 & 3 & 0 \end{pmatrix} =$$

$$= 2(2)(2)\det\begin{pmatrix} 2 & -1/2 \\ 3 & 0 \end{pmatrix} - 2(3)(2)\det\begin{pmatrix} 2 & 0 \\ 3 & -1/2 \end{pmatrix} - 2(3)\left(-\frac{1}{3}\right)\det\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} +$$

$$+ 3(2)\det\begin{pmatrix} 2 & -1/3 \\ 3 & 0 \end{pmatrix} = 8\left(\frac{3}{2}\right) - 2(3)(2)(-1) - 2(3)\left(-\frac{1}{3}\right)(3) + 3(2)(1) = 36 > 0,$$

we obtain that the point A is the *minimum point*.

Since the *bordered hessian matrix* $\overline{H}(B)$ is of *odd order* (order 5) and has the opposite sign of $\overline{H}(A)$, we can conclude that the determinant of $\overline{H}(B)$ has the opposite sign of $\overline{H}(A)$, because for every change of sign in a row or in a column, the determinant changes the sign, and there are five sign changes. Anyway, it could be an useful exercise to expand also the calculation of $\det \overline{H}(B) = -36 < 0$, that is left to the reader, from which we obtain that the point B is the *maximum point*.