MATHEMATICS FOR FINANCE Exam

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Surname ____________________________ **Name** ____________________________

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Exercise 1. Given the canonical basis $\mathcal{B}_{\mathbb{R}^3} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ of the vector space $\mathbb{R}^3$ and the linear application $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ acting on the basis vectors of $\mathbb{R}^3$ according the transformation laws

$$\begin{cases} L(\boldsymbol{e}_1) = 7\boldsymbol{e}_1 + 10\boldsymbol{e}_2 + 9\boldsymbol{e}_3 \\ L(\boldsymbol{e}_2) = 3\boldsymbol{e}_1 + 2\boldsymbol{e}_2 + \boldsymbol{e}_3 \\ L(\boldsymbol{e}_3) = -13\boldsymbol{e}_1 - 14\boldsymbol{e}_2 - 11\boldsymbol{e}_3 \end{cases}$$

1) write the matrix A associated to the linear application L with respect to the given basis;

2) find the subspaces *kernel* and *image* of the linear application L determining a basis for both subspaces.

Let us consider the linear application $\tilde{L} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by the transformation laws of the components

$$\tilde{L}(x, y, z) = (2x - y/2 - z, x + y - z, 2x - y),$$

where in the vector space $\mathbb{R}^3$ the same basis $\mathcal{B}_{\mathbb{R}^3}$ as before is fixed.

3) Write the matrix B associated to the linear application $\tilde{L}$ with respect to the given basis and determine the matrix, denoted by M, associated to the product of linear applications in the order $\tilde{L}L$.

4) Verify whether the matrix M is diagonalizable.

If M is diagonalizable,

5) find the basis vectors with respect to which the matrix M assumes a diagonal form denoted by $\mathcal{D}$ and write the matrix C of the basis change such that $C^{-1}MC = \mathcal{D}$;

6) write the diagonal matrix $\mathcal{D}$ (without performing the matrix multiplication $C^{-1}MC$);

7) in the eigenspace corresponding to the eigenvalue of algebraic multiplicity 2, find an eingenvector orthogonal to the vector $\boldsymbol{v} = (5, -1, -2)$.

Exercise 2. Solve the following Cauchy problem

$$\begin{cases} 9y''(x) + 6y'(x) + y(x) = -3e^{-x/3} + 2xe^{-x/3} \\ y(0) = 1 \\ y'(0) = -1 \end{cases}$$

Exercise 3. Find the optimal points of the function

$$f(x, y, z) = 2x + \frac{1}{2}\,y - \frac{8}{3}\,z$$

subject to the constraints $3xy = 2$ and $4yz = -1$.

Solution

Exercise 1.

1) The matrix A is

$$A = \begin{pmatrix} 7 & 3 & -13 \\ 10 & 2 & -14 \\ 9 & 1 & -11 \end{pmatrix},$$

obtained by writing in columns the coefficients of

$$7\boldsymbol{e}_1 + 10\boldsymbol{e}_2 + 9\boldsymbol{e}_3, \qquad 3\boldsymbol{e}_1 + 2\boldsymbol{e}_2 + \boldsymbol{e}_3, \qquad -13\boldsymbol{e}_1 - 14\boldsymbol{e}_2 - 11\boldsymbol{e}_3.$$

2) The *kernel* of L is the subspace of $\mathbb{R}^3$ containing the vectors $\boldsymbol{v} = (x, y, z)$ such that it yields $L(\boldsymbol{v}) = \boldsymbol{0}$, that is

$$\begin{pmatrix} 7 & 3 & -13 \\ 10 & 2 & -14 \\ 9 & 1 & -11 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which is an algebraic linear system having *rank* 2, because

$$\det \begin{pmatrix} 7 & 3 & -13 \\ 10 & 2 & -14 \\ 9 & 1 & -11 \end{pmatrix} = 0,$$

and the *minor* of order 2

$$\begin{pmatrix} 10 & 2 \\ 9 & 1 \end{pmatrix},$$

highlighted in the matrix A

$$A = \begin{pmatrix} 7 & 3 & -13 \\ \boxed{\begin{matrix} 10 & 2 \\ 9 & 1 \end{matrix}} & & \begin{matrix} -14 \\ -11 \end{matrix} \end{pmatrix},$$

has determinant not equal to zero. From this *minor* we get the system

$$\begin{cases} 10x + 2y = 14t \\ 9x + y = 11t, \end{cases}$$

in which we have given the arbitrary value $z = t$ to the unknown z, that lays outside the *minor* highlighted in the matrix A. The *kernel* has then dimension 1 because this linear system has ∞^1 solutions which are

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} t,$$

from which we get that the *kernel* has basis vector $(1, 2, 1)$. The *image* of L is spanned by all those column vectors having some component contained inside the *minor* highlighted in the matrix A, that is we have the basis of the *image*

$$\mathcal{B}_{Im(L)} = \left\{ \begin{pmatrix} 7 \\ 10 \\ 9 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\},$$

that is the first and second column of A.

3) The matrix B associated to the linear application $\tilde{L}(x, y, z) = (2x - y/2 - z, x + y - z, 2x - y)$ is the matrix

$$B = \begin{pmatrix} 2 & -1/2 & -1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix},$$

because it reproduces the transformation laws of $\tilde{\mathcal{L}}$, that is

$$\tilde{\mathcal{L}}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2&-1/2&-1\\1&1&-1\\2&-1&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2x-y/2-z\\x+y-z\\2x-y\end{pmatrix}.$$

From the matrix B, one gets the matrix M associated to the product of linear applications in the order $\tilde{L}L$

$$M=BA=\begin{pmatrix}2&-1/2&-1\\1&1&-1\\2&-1&0\end{pmatrix}\begin{pmatrix}7&3&-13\\10&2&-14\\9&1&-11\end{pmatrix}=\begin{pmatrix}0&4&-8\\8&4&-16\\4&4&-12\end{pmatrix}.$$

4,5) In order to verify whether the matrix M, which is an *endomorphism* of $\mathbb{R}^3$, is *diagonalizable*, we have to extablish whether there exists a basis of the vector space $\mathbb{R}^3$ consisting of three eigenvectors of M, that is we have to verify, in other words, whether there exist three *linearly independent* eigenvectors of M.

The *characteristic polynomial* of M is

$$\det\begin{pmatrix}-\lambda&4&-8\\8&4-\lambda&-16\\4&4&-12-\lambda\end{pmatrix}=-\lambda[(\lambda+12)(\lambda-4)+64]-4[64-8(\lambda+12)]-8[32-4(4-\lambda)]=$$

$$=-\lambda\left(\lambda^2+8\lambda+16\right)+4(8\lambda+32)-8(4\lambda+16)=-\lambda(\lambda^2+8\lambda+16)=-\lambda(\lambda+4)^2,$$

whose zeros are the *simple*[1] eigenvalue $\lambda=0$ and the eigenvalue $\lambda=-4$, having *algebraic multiplicity* 2.

To the *simple* eigenvalue $\lambda=0$ we associate the linear system $(M-0\mathbb{I})\boldsymbol{u}=\mathbf{0}$, that is

$$\begin{pmatrix}0&4&-8\\8&4&-16\\4&4&-12\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix},$$

whose ∞^1 solutions, by virtue of the *minor* of order 2

$$\begin{pmatrix}\boxed{\begin{matrix}0&4\\8&4\end{matrix}}&\begin{matrix}-8\\-16\end{matrix}\\\begin{matrix}4&4\end{matrix}&-12\end{pmatrix},$$

are $x=t, y=2t, z=t$, from which we get the eigenvector $\boldsymbol{u}=(1,2,1)$, satisfying effectively the equality

$$\begin{pmatrix}0&4&-8\\8&4&-16\\4&4&-12\end{pmatrix}\begin{pmatrix}1\\2\\1\end{pmatrix}=0\begin{pmatrix}1\\2\\1\end{pmatrix},\qquad\text{that is}\qquad M\boldsymbol{u}=0\boldsymbol{u}.$$

To the eigenvalue $\lambda=-4$, we associate the linear system $[M-(-4)\mathbb{I}]\boldsymbol{w}=\mathbf{0}$, that is

$$\begin{pmatrix}4&4&-8\\8&8&-16\\4&4&-8\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix},$$

whose ∞^2 solutions, by virtue of the *minor* of order 1

$$\begin{pmatrix}\boxed{4}&4&-8\\8&8&-16\\4&4&-8\end{pmatrix}$$

[1]An eigenvalue λ of a matrix is called *simple eigenvalue* if its *algebraic multiplicity* is 1.

are

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \alpha + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \beta, \tag{1}$$

from which we get the two eigenvectors $\boldsymbol{w}_1 = (-1, 1, 0)$ and $\boldsymbol{w}_2 = (2, 0, 1)$, satisfying the equalities

$$\begin{pmatrix} 0 & 4 & -8 \\ 8 & 4 & -16 \\ 4 & 4 & -12 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = -4 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} 0 & 4 & -8 \\ 8 & 4 & -16 \\ 4 & 4 & -12 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = -4 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix},$$

that is $M\boldsymbol{w}_1 = -4\boldsymbol{w}_1$ and $M\boldsymbol{w}_2 = -4\boldsymbol{w}_2$.

Since the set $\mathcal{B} = \{\boldsymbol{u}, \boldsymbol{w}_1, \boldsymbol{w}_2\}$, containing three eigenvectors of the matrix M, is linearly independent, we conclude that the set $\mathcal{B}$ is a basis of the vector space $\mathbb{R}^3$, and then that the matrix M is *diagonalizable*.

The matrix C of the basis change to the basis of the eigenvectors, with respect to which M assumes diagonal form, is then the one whose columns are the eigenvectors, that is

$$C = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

6) Since we have written the eigenvectors in the matrix C in the sequence corresponding to the eigenvalues in the order $\lambda = 0, -2, -2$, respectively, it follows that the diagonal matrix $\mathcal{D}$, associated to M, is

$$\mathcal{D} = C^{-1}MC = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

7) The eigenspace associated to the eigenvalue of algebraic multiplicity 2 is the one corresponding to $\lambda = -4$, spanned by the two eigenvectors $\boldsymbol{w}_1, \boldsymbol{w}_2$, that we denote by $\mathbb{E}_{-4}$. The vectors of this subspace have the form (1), and the vector of this subspace, orthogonal to the given vector $\boldsymbol{v}$, is the vector $\boldsymbol{w} = (-\alpha + 2\beta, \alpha, \beta)$ such that the *scalar product* $(\boldsymbol{v}, \boldsymbol{w})$ vanishes, that is

$$(\boldsymbol{v}, \boldsymbol{w}) = (5, -1, -2) \cdot (-\alpha + 2\beta, \alpha, \beta) = 0,$$

from which we get the relation $3\alpha - 4\beta = 0$. By choosing the particular solution given by $\alpha = 4, \beta = 3$, we finally obtain the particular vector $\boldsymbol{w} = (2, 4, 3)$ belonging to $\mathbb{E}_{-4}$ and orthogonal to the given vector $\boldsymbol{v} = (5, -1, -2)$.

Exercise 2.

The homogeneous equation associated to the given equation is

$$9y''(x) + 6y'(x) + y(x) = 0,$$

to which the algebraic equation

$$9\lambda^2 + 6\lambda + 1 = 0$$

corresponds, having the solution $\lambda = -1/3$ with algebraic multiplicity 2. The solution, that we denote by $y_0(x)$, of the homogeneous equation is then

$$y_0(x) = Ae^{-x/3} + Bxe^{-x/3},$$

and since the right-hand side of the given equation is $(2x - 3)e^{-x/3}$, that is the product of a polynomial of first degree times the exponential $e^{-x/3}$, we write the *particular solution* $y_p(x)$ in the same form

$$y_p(x) = (ax + b)e^{-x/3}.$$

Since this $y_p(x)$ is similar to the solution of the homogeneous equation, we multiply $y_p(x)$ times x, and we obtain the new *particular solution*

$$y_p(x) = (ax^2 + bx)e^{-x/3},$$

whose term $bxe^{-x/3}$ is similar to $Bxe^{-x/3}$ of the solution of the homogeneous equation. We have then to multiply for another factor x in such a way that the *particular solution* $y_p(x)$ assumes the final form

$$y_p(x) = (ax^3 + bx^2)e^{-x/3},$$

whose derivatives are

$$y_p'(x) = \left(-\frac{a}{3}\,x^3 + 3ax^2 - \frac{b}{3}\,x^2 + 2bx\right)e^{-x/3},$$

$$y_p''(x) = \left(\frac{a}{9}\,x^3 - 2ax^2 + \frac{b}{9}\,x^2 + 6ax - \frac{4}{3}\,bx + 2b\right)e^{-x/3}.$$

By inserting $y_p(x), y_p'(x), y_p''(x)$ into the given equation, we get the equality

$$9\left(\frac{a}{9}\,x^3 - 2ax^2 + \frac{b}{9}\,x^2 + 6ax - \frac{4}{3}\,bx + 2b\right)e^{-x/3} +$$

$$+6\left(-\frac{a}{3}\,x^3 + 3ax^2 - \frac{b}{3}\,x^2 + 2bx\right)e^{-x/3} + (ax^3 + bx^2)e^{-x/3} = (2x-3)e^{-x/3},$$

that, after the semplifications

$$\left(\cancel{ax^3} - \cancel{18ax^2} + \cancel{bx^2} + 54ax - \cancel{12bx} + 18b\right)e^{-x/3} +$$

$$+\left(-\cancel{2ax^3} + \cancel{18ax^2} - \cancel{2bx^2} + \cancel{12bx}\right)e^{-x/3} + (\cancel{ax^3} + \cancel{bx^2})e^{-x/3} = (2x-3)e^{-x/3},$$

becomes

$$(54ax + 18b)e^{-x/3} = (2x-3)e^{-x/3},$$

from which we obtain the two equations $54a = 2, 18b = -3$, and then $a = 1/27, b = -1/6$.

The solution of the given differential equation is then

$$y(x) = Ae^{-x/3} + Bxe^{-x/3} + \frac{1}{27}\,x^3e^{-x/3} - \frac{1}{6}\,x^2e^{-x/3},$$

whose first derivative is

$$y'(x) = -\frac{A}{3}\,e^{-x/3} + Be^{-x/3} - \frac{B}{3}\,xe^{-x/3} + \frac{1}{9}\,x^2e^x - \frac{1}{81}\,x^3e^{-x/3} - \frac{1}{3}\,xe^{-x/3} + \frac{1}{18}\,x^2e^x,$$

from which, by imposing the *initial conditions* $y(0) = 1, y'(0) = -1$ of the *Cauchy problem*, the system

$$\begin{cases} A & = & 1 \\ -A/3 + B & = & -1 \end{cases}$$

follows, having solution $A = 1, B = -2/3$. The solution of the given *Cauchy problem* is then

$$y(x) = e^{-x/3} - \frac{2}{3}\,xe^{-x/3} + \frac{1}{27}\,x^3e^{-x/3} - \frac{1}{6}\,x^2e^{-x/3}.$$

Exercise 3. The *Lagrangian function* $\mathcal{L}$ associated to the given optimization problem is

$$\mathcal{L}(e, y, z; \lambda, \mu) = 2x + \frac{1}{2}\,y - \frac{8}{3}\,z + \lambda(3xy - 2) + \mu(4yz + 1),$$

from which the *first order conditions*

$$\begin{cases} 2+3\lambda y=0 \\ 1/2+3\lambda x+4\mu z=0 \\ -8/3+4\mu y=0 \\ 3xy=2 \\ 4yz=-1. \end{cases}$$

From the first, third, fourth, fifth equation, we get

$$\lambda=-2/(3y), \qquad \mu=2/(3y), \qquad x=2/(3y), \qquad z=-1/(4y),$$

respectively, that, inserted into the second equation, give

$$\frac{1}{2}+3\left(-\frac{2}{3y}\right)\left(\frac{2}{3y}\right)+4\left(\frac{2}{3y}\right)\left(-\frac{1}{4y}\right)=0 \qquad \Longrightarrow \qquad \frac{y^2-4}{2y^2}=0,$$

where $y \neq 0$ because $y=0$ is not consistent with the constraints. From $y^2-4=0$, we get $y=\pm 2$ and then the *optimal points*

$$A=\left(\frac{1}{3},2,-\frac{1}{8};-\frac{1}{3},\frac{1}{3}\right) \qquad \text{and} \qquad B=\left(-\frac{1}{3},-2,\frac{1}{8};\frac{1}{3},-\frac{1}{3}\right).$$

The *bordered hessian matrix* is

$$\overline{H}(x,y,z;\lambda,\mu)=\begin{pmatrix} 0 & 0 & 3y & 3x & 0 \\ 0 & 0 & 0 & 4z & 4y \\ 3y & 0 & 0 & 3\lambda & 0 \\ 3x & 4z & 3\lambda & 0 & 4\mu \\ 0 & 4y & 0 & 4\mu & 0 \end{pmatrix},$$

from which we get

$$\overline{H}(A)=\begin{pmatrix} 0 & 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & -1/2 & 8 \\ 6 & 0 & 0 & -1 & 0 \\ 1 & -1/2 & -1 & 0 & 4/3 \\ 0 & 8 & 0 & 4/3 & 0 \end{pmatrix} \qquad \text{and} \qquad \overline{H}(B)=\begin{pmatrix} 0 & 0 & -6 & -1 & 0 \\ 0 & 0 & 0 & 1/2 & -8 \\ -6 & 0 & 0 & 1 & 0 \\ -1 & 1/2 & 1 & 0 & -4/3 \\ 0 & -8 & 0 & -4/3 & 0 \end{pmatrix}.$$

From

$$\det\overline{H}(A)=6\det\begin{pmatrix} 0 & 0 & -1/2 & 8 \\ 6 & 0 & -1 & 0 \\ 1 & -1/2 & 0 & 4/3 \\ 0 & 8 & 4/3 & 0 \end{pmatrix}-\det\begin{pmatrix} 0 & 0 & 0 & 8 \\ 6 & 0 & 0 & 0 \\ 1 & -1/2 & -1 & 4/3 \\ 0 & 8 & 0 & 0 \end{pmatrix}=$$

$$=6\left(-\frac{1}{2}\right)\det\begin{pmatrix} 6 & 0 & 0 \\ 1 & -1/2 & 4/3 \\ 0 & 8 & 0 \end{pmatrix}-6(8)\det\begin{pmatrix} 6 & 0 & -1 \\ 1 & -1/2 & 0 \\ 0 & 8 & 4/3 \end{pmatrix}+8\det\begin{pmatrix} 6 & 0 & 0 \\ 1 & -1/2 & -1 \\ 0 & 8 & 0 \end{pmatrix}=$$

$$=6\left(-\frac{1}{2}\right)(6)\det\begin{pmatrix} -1/2 & 4/3 \\ 8 & 0 \end{pmatrix}-6(8)(6)\det\begin{pmatrix} -1/2 & 0 \\ 8 & 4/3 \end{pmatrix}+6(8)\det\begin{pmatrix} 1 & -1/2 \\ 0 & 8 \end{pmatrix}+$$

$$+8(6)\det\begin{pmatrix} -1/2 & -1 \\ 8 & 0 \end{pmatrix}=-18\left(-\frac{32}{2}\right)-6(8)(6)(-2/3)+6(8)(8)+8(6)(8)=1248>0,$$

we obtain that the point A is the *minimum point*.

Since the *bordered hessian matrix* $\overline{H}(B)$ is of *odd order* (order 5) and has the opposite sign of $\overline{H}(A)$, we can conclude that the determinant of $\overline{H}(B)$ has the opposite sign of $\overline{H}(A)$, because for every change of sign in a row or in a column, the determinant changes the sign, and there are five sign changes. Anyway, it could be an useful exercise to expand also the calculation of $\det \overline{H}(B) = -1248 < 0$, that is left to you, from which we obtain that the point B is the *maximum point*.