Exercises on Linear Algebra

Exercise 1. Given the linear system

$$\begin{cases} -5x + ky + z = 2 \\ 7x + y - 6z = -8 \\ -3x - 7y + kz = k \end{cases}$$

1) determine the number of solutions for every value of k;
2) determine the explicit solutions for every value of k.

Exercise 2. Given the linear system

$$\begin{cases} -3y + kx = k - 7x \\ 4x + 2ky - 3k = 5 - 4y \end{cases}$$

1) determine the number of solutions for every value of k;
2) determine the explicit solutions for every value of k.

Exercise 3. Check whether the following set $\mathcal{I}$ of three vectors belonging to the vector space $\mathbb{R}^4$ is *linearly dependent* or *linearly independent*

$$\mathcal{I} = \left\{ \boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 2 \\ -3 \\ -2 \\ 3 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 3 \\ 4 \\ -3 \end{pmatrix} \right\}$$

and if the set would be *linearly dependent*, write a vector as linear combination of two vectors.

Exercise 4. Check whether the following set $\mathcal{I}$ of three vectors belonging to the vector space $\mathbb{R}^4$ is *linearly dependent* or *linearly independent*

$$\mathcal{I} = \left\{ \boldsymbol{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 3 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 5 \\ 3 \\ 1 \\ 8 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} 1 \\ 5 \\ 1 \\ 2 \end{pmatrix} \right\}$$

and if the set would be *linearly dependent*, write a vector as linear combination of two vectors.

Exercise 5. Verify whether the following set S of $\mathbb{R}^3$ is a *vector subspace* or not

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 \text{ such that } 2x - y = 0 \right\}.$$

If it is, find its dimension and a basis.

Exercise 6. Find the *rank* of the following set $\mathcal{J}$ of four vectors

$$\mathcal{J} = \left\{ \boldsymbol{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 3 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} -3 \\ -4 \\ -1 \\ -5 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} 1 \\ 5 \\ 1 \\ 2 \end{pmatrix}, \boldsymbol{v}_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 3 \end{pmatrix} \right\}$$

and if the *rank* of the set would be less than 4, find a *linearly independent* set $\mathcal{J}_1$ which is a proper subset of $\mathcal{J}$ such that $\mathcal{J}_1$ has as many vectors as the *rank* is.

Exercise 7. Find the subspace *intersection* of the two subspaces $\mathcal{U}, \mathcal{W}$ of $\mathbb{R}^5$ spanned by its own basis vectors

$$\mathcal{U} = \left\{ \boldsymbol{u}_1 = \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 1 \\ -2 \end{pmatrix}, \quad \boldsymbol{u}_3 = \begin{pmatrix} -2 \\ 1 \\ -4 \\ -1 \\ 0 \end{pmatrix} \right\},$$

$$\mathcal{W} = \left\{ \boldsymbol{w}_1 = \begin{pmatrix} 0 \\ 1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \quad \boldsymbol{w}_2 = \begin{pmatrix} 5 \\ 2 \\ -2 \\ 1 \\ -6 \end{pmatrix} \right\}$$

Solution of the exercise 7.

First method. If we write the generic vector of the subspaces in the parametric form of linear combination of the basis vectors, we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \alpha_3 \boldsymbol{u}_3 = \alpha_1 \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ 0 \\ 2 \\ 1 \\ -2 \end{pmatrix} + \alpha_3 \begin{pmatrix} -2 \\ 1 \\ -4 \\ -1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \beta_1 \boldsymbol{w}_1 + \beta_2 \boldsymbol{w}_2 = \beta_1 \begin{pmatrix} 0 \\ 1 \\ -3 \\ 2 \\ 0 \end{pmatrix} + \beta_2 \begin{pmatrix} 5 \\ 2 \\ -2 \\ 1 \\ -6 \end{pmatrix}.$$

The subspace intersection of $\mathcal{U}, \mathcal{W}$ consists of the vectors belonging to both subspaces, simultaneously, and is then spanned by those vectors such that the following equality holds

$$\alpha_1 \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ 0 \\ 2 \\ 1 \\ -2 \end{pmatrix} + \alpha_3 \begin{pmatrix} -2 \\ 1 \\ -4 \\ -1 \\ 0 \end{pmatrix} = \beta_1 \begin{pmatrix} 0 \\ 1 \\ -3 \\ 2 \\ 0 \end{pmatrix} + \beta_2 \begin{pmatrix} 5 \\ 2 \\ -2 \\ 1 \\ -6 \end{pmatrix},$$

that is

$$\begin{cases} 2\alpha_1 + 3\alpha_2 - 2\alpha_3 - 5\beta_2 = 0 \\ -3\alpha_1 + \alpha_3 - \beta_1 - 2\beta_2 = 0 \\ 2\alpha_2 - 4\alpha_3 + 3\beta_1 + 2\beta_2 = 0 \\ \alpha_1 + \alpha_2 - \alpha_3 - 2\beta_1 - \beta_2 = 0 \\ \alpha_1 - 2\alpha_2 + 6\beta_2 = 0. \end{cases}$$

If this homogeneous system has unique solution, that is the trivial solution, this means that the subspace intersection of $\mathcal{U}, \mathcal{W}$ consists of the null vector, only. If the solution of this homogeneous system are ∞^d, this means that the dimension of the subspace intersection of $\mathcal{U}, \mathcal{W}$ is d, and I leave to the reader as a simple exercise to find the parametric and cartesian forms of the subspace intersection of $\mathcal{U}, \mathcal{W}$.