MATHEMATICS FOR FINANCE Exam

January 2024, the 16th

Surname ______________________________ **Name** ______________________________

ID Number ________________________

Exercise 1. Given the canonical basis $\mathcal{B}_{\mathbb{R}^4} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ of the vector spaces $\mathbb{R}^4$, and the linear application $L : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ acting on the basis vectors of $\mathbb{R}^4$ according the transformation laws

$$\begin{cases} L(\boldsymbol{e}_1) = 2\boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_3 + 5\boldsymbol{e}_4 \\ L(\boldsymbol{e}_2) = \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_4 \\ L(\boldsymbol{e}_3) = 2\boldsymbol{e}_1 + 4\boldsymbol{e}_4 \\ L(\boldsymbol{e}_4) = 2\boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_3 + 5\boldsymbol{e}_4, \end{cases}$$

1) write the matrix A associated to the linear application L with respect to the given basis;

2) find the subspaces *kernel* and *image* of the linear application L determining their dimension and a basis for both subspaces;

3) find the *orthogonal projection* of the vector $\boldsymbol{u} = (-2, 1, 1, 1)$ on the subspace *image* of L.

Let us consider the linear application $\tilde{L} : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ defined by the transformation laws of the components

$$\tilde{L}(x_1, x_2, x_3, x_4) = (x_1 - x_3, x_2 - x_3, x_1 - x_4, -x_1),$$

where in the vector space $\mathbb{R}^4$ the same basis $\mathcal{B}_{\mathbb{R}^4}$ is fixed as before.

4) Write the matrix B associated to the linear application $\tilde{L}$ with respect to the given basis and determine the matrix, denoted by M, associated to the composition of linear applications $L \circ \tilde{L}$ (matrix product AB).

5) Verify whether the matrix M is diagonalizable.

If M is diagonalizable,

6) find the basis vectors with respect to which the matrix M assumes a diagonal form denoted by $\mathcal{D}$ and write the matrix C of the basis change such that $C^{-1}MC = \mathcal{D}$;

7) write the diagonal matrix $\mathcal{D}$ (without performing the matrix multiplication $C^{-1}MC$);

8) in the eigenspace of the matrix M corresponding to the eigenvalue having algebraic multiplicity 2, find an eingenvector $\boldsymbol{v}$ of M which is orthogonal to the vector $\boldsymbol{w} = (0, 0, 2, 1)$;

9) find a basis of the subspace *orthogonal complement* of the eigenspace of the matrix M corresponding to the eigenvalue having algebraic multiplicity 2.

Exercise 2. Solve the following Cauchy problem

$$\begin{cases} 4y''(x) + 4y'(x) + y(x) = 3e^{-x/2} \\ y(0) = -2 \\ y'(0) = 2 \end{cases}$$

Exercise 3. Find the optimal points of the function

$$f(x, y, z) = x + 2y - 3z$$

subject to the constraint $2x^2 + y^2 - z^2 + x + z = -3$

Solution of the exam of the day January 2024, the 16th

Exercise 1.

1) The matrix A is

$$A = \begin{pmatrix} 2 & 1 & 2 & 2 \\ -1 & 1 & 0 & -1 \\ -1 & 0 & 0 & -1 \\ 5 & 1 & 4 & 5 \end{pmatrix},$$

obtained by writing in columns the coefficients of

$$2\boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_3 + 5\boldsymbol{e}_4\,, \qquad \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_4\,, \qquad 2\boldsymbol{e}_1 + 4\boldsymbol{e}_4\,, \qquad 2\boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_3 + 5\boldsymbol{e}_4\,.$$

2) The *kernel* of L is the subspace of $\mathbb{R}^4$ containing the vectors $\boldsymbol{k} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ such that the equality $L(\boldsymbol{k}) = \boldsymbol{0}$ holds, that is

$$\begin{pmatrix} 2 & 1 & 2 & 2 \\ -1 & 1 & 0 & -1 \\ -1 & 0 & 0 & -1 \\ 5 & 1 & 4 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which is an algebraic linear system having *rank* 3, because

$$\det\begin{pmatrix} 2 & 1 & 2 & 2 \\ -1 & 1 & 0 & -1 \\ -1 & 0 & 0 & -1 \\ 5 & 1 & 4 & 5 \end{pmatrix} = 2\det\begin{pmatrix} -1 & 1 & -1 \\ -1 & 0 & -1 \\ 5 & 1 & 5 \end{pmatrix} - 4\det\begin{pmatrix} 2 & 1 & 2 \\ -1 & 1 & -1 \\ -1 & 0 & -1 \end{pmatrix} =$$

$$= 2\left[-\det\begin{pmatrix} -1 & -1 \\ 5 & 5 \end{pmatrix} - \det\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\right] - 4\left[-\det\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} - \det\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}\right] = 0$$

and the *minor* of order 3

$$\mathfrak{M} = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix},$$

highlighted in the matrix A as shown

$$A = \left(\begin{array}{c|ccc|} 2 & 1 & 2 & 2 \\ -1 & 1 & 0 & -1 \\ -1 & 0 & 0 & -1 \\ \hline 5 & 1 & 4 & 5 \end{array}\right),$$

has determinant

$$\det\mathfrak{M} = \det\begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix} = -\det\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = 2 \neq 0.$$

By virtue of this *minor* $\mathfrak{M}$, we can extract the system

$$\begin{cases} x_2 + 2x_3 + 2x_4 = -2t \\ x_2 - x_4 = t \\ -x_4 = t \end{cases}$$

where we have given the arbitrary value $x_1 = t$ to the unknown x_1 that lays out of the *minor* $\mathfrak{M}$ highlighted in the matrix A. The *kernel* has then dimension 1 because this linear system has the $\infty^{4-3} = \infty^1$ solutions

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} t,$$

from which we get that a basis vector of the *kernel* is the vector $\boldsymbol{k} = (1, 0, 0, -1)$, as it can be verified through

$$L(\boldsymbol{k}) = A\boldsymbol{k} = \begin{pmatrix} 2 & 1 & 2 & 2 \\ -1 & 1 & 0 & -1 \\ -1 & 0 & 0 & -1 \\ 5 & 1 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The *image* of L is spanned by all those column vectors having some component contained inside the *minor* highlighted in the matrix A, that is we have the basis of the *image*

$$\mathcal{B}_{Im(L)} = \boldsymbol{w}_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{w}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \quad \boldsymbol{w}_3 = \begin{pmatrix} 2 \\ -1 \\ -1 \\ 5 \end{pmatrix} \right\},$$

that is the second, third, and fourth column of A, where the third column for $\boldsymbol{w}_2$ has been divided by 2.

3) The *orthogonal projection* of the vector $\boldsymbol{u}$ on the *image* of L is the vector, that we denote by $\boldsymbol{p}$ belonging to the *image*, such that it yields

$$\langle \boldsymbol{u} - \boldsymbol{p}\,,\, \boldsymbol{w}_1 \rangle = 0, \qquad \langle \boldsymbol{u} - \boldsymbol{p}\,,\, \boldsymbol{w}_2 \rangle = 0, \qquad \langle \boldsymbol{u} - \boldsymbol{p}\,,\, \boldsymbol{w}_3 \rangle = 0. \tag{1}$$

By expanding the vector $\boldsymbol{p} \in Im(L)$ as linear combination of the basis vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ of the *image*, that is

$$\boldsymbol{p} = \alpha \boldsymbol{w}_1 + \beta \boldsymbol{w}_2 + \gamma \boldsymbol{w}_3\,, \tag{2}$$

we have

$$\boldsymbol{u} - \boldsymbol{p} = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \beta \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} - \gamma \begin{pmatrix} 2 \\ -1 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 - \alpha - \beta - 2\gamma \\ 1 - \alpha + \gamma \\ 1 + \gamma \\ 1 - \alpha - 2\beta - 5\gamma \end{pmatrix},$$

by virtue of which the three equations (1) assume the form of the linear system

$$\begin{cases} -3\alpha - 3\beta - 6\gamma = 0 \\ -3\alpha - 5\beta - 12\gamma = 0 \\ 6\alpha + 12\beta + 31\gamma = -1. \end{cases}$$

The sum of the three equations and the subtraction of the first two equations give the two equations

$$\begin{cases} 4\beta + 13\gamma = -1 \\ \beta + 3\gamma = 0, \end{cases}$$

respectively, from which we get

$$\alpha = -1, \qquad \beta = 3, \qquad \gamma = -1$$

and then, from (2), the *orthogonal projection* $\boldsymbol{p} = (0, 0, 1, 0)$.

4) The matrix B associated to the linear application $\tilde{L}(x_1, x_2, x_3, x_4) = (x_1 - x_3, x_2 - x_3, x_1 - x_4, -x_1)$ is the matrix

$$B = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

because it reproduces the given transformation laws of $\tilde{L}$, that is

$$\tilde{L}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 - x_3 \\ x_2 - x_3 \\ x_1 - x_4 \\ -x_1 \end{pmatrix}.$$

From the matrix B, one gets the matrix M associated to the product of linear applications in the order $L\tilde{L}$

$$M = AB = \begin{pmatrix} 2 & 1 & 2 & 2 \\ -1 & 1 & 0 & -1 \\ -1 & 0 & 0 & -1 \\ 5 & 1 & 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 1 & -6 & -4 \end{pmatrix}.$$

5,6) In order to verify whether the matrix M, which is an *endomorphism* of $\mathbb{R}^4$, is *diagonalizable*, we have to extablish whether there exists a basis of the vector space $\mathbb{R}^4$ consisting of four eigenvectors of M, that is we have to verify, in other words, whether there exist four *linearly independent* eigenvectors of M, which are *basis eigenvectors* of their corresponding eigenspaces, denoted by $\mathbb{E}(\lambda_i)$, where λ_i represents an eigenvalue of M.

Due to the expansion of the determinant according to the second row, the *characteristic polynomial* of M is

$$\det(M - \lambda\mathbb{I}) = \det\begin{pmatrix} 2-\lambda & 1 & -3 & -2 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 4 & 1 & -6 & -4-\lambda \end{pmatrix} = (1-\lambda)\det\begin{pmatrix} 2-\lambda & -3 & -2 \\ 0 & 1-\lambda & 0 \\ 4 & -6 & -4-\lambda \end{pmatrix} =$$

$$= (1-\lambda)(1-\lambda)\det\begin{pmatrix} 2-\lambda & -2 \\ 4 & -4-\lambda \end{pmatrix} = \lambda(\lambda-1)^2(\lambda+2),$$

whose zeros are:

- the *simple*[1] eigenvalues $\lambda = 0$ and $\lambda = -2$,
- the eigenvalue $\lambda = 1$, having *algebraic multiplicity* 2.

To the *simple* eigenvalue $\lambda = 0$ we associate the linear system $(M - 0\mathbb{I})\boldsymbol{u} = \mathbf{0}$, that is

$$\begin{pmatrix} 2 & 1 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 1 & -6 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

having *rank* 3 by virtue of the following *minor matrix* of order 3 highlighted in M

$$M = \left(\begin{array}{|ccc|c} \hline 2 & 1 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 4 & 1 & -6 & -4 \end{array}\right),$$

[1]We remind that an eigenvalue λ of a matrix is called *simple eigenvalue* if its *algebraic multiplicity* is 1.

from which it follows that the system has ∞^1 solutions, and the eigenspace $\mathbb{E}(0)$ has *dimension* 1.

By virtue of the highlighted *minor matrix*, we put $x_4 = t$ and solve $2x_1 + x_2 - 3x_3 = 2t, x_2 = 0, x_3 = 0$, from which we get $x_1 = t$ and then the first eigenvector $\boldsymbol{u}_{(0)} = (1, 0, 0, 1)$ as basis eigenvector of $\mathbb{E}(0)$, satisfying effectively the equality

$$\begin{pmatrix} 2 & 1 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 1 & -6 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \text{that is} \qquad M\boldsymbol{u}_{(0)} = 0\boldsymbol{u}_{(0)}.$$

To the *simple* eigenvalue $\lambda = -2$, we associate the linear system $[M - (-2)\mathbb{I}]\boldsymbol{u} = \mathbf{0}$, that is

$$\begin{pmatrix} 4 & 1 & -3 & -2 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 4 & 1 & -6 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

having *rank* 3 by virtue of the following *minor matrix* of order 3 highlighted in $M + 2\mathbb{I}$

$$M + 2\mathbb{I} = \begin{pmatrix} \boxed{\begin{matrix} 4 & 1 & -3 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{matrix}} & \begin{matrix} -2 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 4 & 1 & -6 \end{matrix} & -2 \end{pmatrix},$$

from which it follows that the system has ∞^1 solutions, and the eigenspace $\mathbb{E}(-2)$ has *dimension* 1.

By virtue of the highlighted *minor matrix*, we put $x_4 = t$ and solve $4x_1 + x_2 - 3x_3 = 2t, x_2 = 0, x_3 = 0$, from which we get $x_1 = t/2$ and then, by eliminating the fraction, the second eigenvector $\boldsymbol{u}_{(-2)} = (1, 0, 0, 2)$ as basis eigenvector of the eigenspace $\mathbb{E}(-2)$, satisfying effectively the equality

$$\begin{pmatrix} 2 & 1 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 1 & -6 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \qquad \text{that is} \qquad M\boldsymbol{u}_{(-2)} = -2\boldsymbol{u}_{(-2)}.$$

To the eigenvalue $\lambda = 1$, having *algebraic multiplicity* 2, we associate the system $(M - \mathbb{I})\boldsymbol{u} = \mathbf{0}$, that is

$$\begin{pmatrix} 1 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 1 & -6 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

having *rank* 2 by virtue of the following *minor matrix* of order 2 highlighted in $M - \mathbb{I}$

$$M - \mathbb{I} = \begin{pmatrix} \boxed{1 \quad 1} & -3 & -2 \\ 0 \quad 0 & 0 & 0 \\ 0 \quad 0 & 0 & 0 \\ \boxed{4 \quad 1} & -6 & -5 \end{pmatrix},$$

from which it follows that the system has ∞^2 solutions, and the eigenspace $\mathbb{E}(1)$ has *dimension* 2. By virtue of the highlighted *minor matrix*, we put $x_3 = \alpha, x_4 = \beta$ and solve $x_1 + x_2 = 3\alpha + 2\beta, 4x_1 + x_2 = 6\alpha + 5\beta$, from which, by subtracting, we get $3x_1 = 3\alpha + 3\beta$ and then the last two eigenvectors

$$\boldsymbol{u}_{(1)}^{(a)} = (1, 2, 1, 0) \qquad \text{and} \qquad \boldsymbol{u}_{(1)}^{(b)} = (1, 1, 0, 1)$$

as basis eigenvectors of the eigenspace $\mathbb{E}(1)$, satisfying effectively the equalities

$$\begin{pmatrix} 2 & 1 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 1 & -6 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 2 & 1 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 1 & -6 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix},$$

that is $M\boldsymbol{u}_{(1)}^{(a)} = \boldsymbol{u}_{(1)}^{(a)}$ and $M\boldsymbol{u}_{(1)}^{(b)} = \boldsymbol{u}_{(1)}^{(b)}$.

Since the set $\mathcal{B} = \{\boldsymbol{u}_{(0)}, \boldsymbol{u}_{(-2)}, \boldsymbol{u}_{(1)}^{(a)}, \boldsymbol{u}_{(1)}^{(b)}\}$, containing the four eigenvectors of the matrix M, is linearly independent, we conclude that the set $\mathcal{B}$ is a basis of the vector space $\mathbb{R}^4$, and the matrix M is *diagonalizable*.

The matrix C describing the basis change from the *initial basis* to the basis of the eigenvectors, with respect to which M assumes *diagonal form*, is then the one whose columns are the four eigenvectors, that is

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}.$$

7) Since we have written the eigenvectors in the matrix C in the sequence corresponding to the eigenvalues in the order $\lambda = 0, -2, 1, 1$, respectively, it follows that the diagonal matrix $\mathcal{D}$, associated to M, is

$$\mathcal{D} = C^{-1}MC = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

8) The eigenspace associated to the eigenvalue having algebraic multiplicity 2 is $\mathbb{E}(1)$, corresponding to the eigenvalue $\lambda = 1$, spanned by the two eigenvectors $\boldsymbol{u}_{(1)}^{(a)}, \boldsymbol{u}_{(1)}^{(b)}$. The vectors of this subspace have the form

$$(x_1, x_2, x_3, x_4) = (\alpha + \beta, 2\alpha + \beta, \alpha, \beta),$$

and the vector $\boldsymbol{v}$ of this subspace, orthogonal to the given vector $\boldsymbol{w} = (0, 0, 2, 1)$, is the vector

$$\boldsymbol{v} = (\alpha + \beta, 2\alpha + \beta, \alpha, \beta)$$

such that the *scalar product* $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$ vanishes, that is the equality

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \langle\, (\alpha + \beta, 2\alpha + \beta, \alpha, \beta)\,,\, (0, 0, 2, 1)\,\rangle = 0$$

holds, from which we get the relation $2\alpha + \beta = 0$. By choosing the particular solution given by $\alpha = -1, \beta = 2$, we finally obtain the particular vector $\boldsymbol{v} = (1, 0, -1, 2)$ belonging to the eigenspace $\mathbb{E}(1)$ and orthogonal to the given vector $\boldsymbol{w} = (0, 0, 2, 1)$.

9) The eigenspace $\mathbb{E}(1)$ associated to the eigenvalue having algebraic multiplicity 2 is spanned by the two eigenvectors $\boldsymbol{u}_{(1)}^{(a)}, \boldsymbol{u}_{(1)}^{(b)}$ and its *orthogonal complement* consists of all vectors $\boldsymbol{v}^{\perp} = (y_1, y_2, y_3, y_4)$ orthogonal to every vector of $\mathbb{E}(1)$ itself. By virtue of the *theorem of the orthogonal complement*, it is actually sufficient that the vectors $\boldsymbol{v}^{\perp} = (y_1, y_2, y_3, y_4)$ to be orthogonal to the basis eigenvectors $\boldsymbol{u}_{(1)}^{(a)}, \boldsymbol{u}_{(1)}^{(b)}$ of $\mathbb{E}(1)$, only.

Therefore, we impose the *orthogonality conditions*

$$\left\langle (y_1, y_2, y_3, y_4)\,,\, \boldsymbol{u}_{(1)}^{(a)} \right\rangle = 0 \qquad \text{and} \qquad \left\langle (y_1, y_2, y_3, y_4)\,,\, \boldsymbol{u}_{(1)}^{(b)} \right\rangle = 0,$$

which are equivalent to the linear system having *rank* 2 and 4 unknowns

$$\begin{cases} y_1 + 2y_2 + y_3 = 0 \\ y_1 + y_2 + y_4 = 0, \end{cases}$$

from which we extract the system (already uncoupled) corresponding to the unknowns y_3, y_4

$$\begin{cases} y_3 = -y_1 - 2y_2 \\ y_4 = -y_1 - y_2 \,. \end{cases}$$

Since this system has ∞^2 solutions having the vector form $(y_1, y_2, y_3, y_4) = (\alpha, \beta, -\alpha-2\beta, -\alpha-\beta)$, we can conclude that the *basis vectors* of the *orthogonal complement* of the eigenspace $\mathbb{E}(1)$ are

$$\boldsymbol{z}_1 = (1, 0, -1, -1) \qquad \text{and} \qquad \boldsymbol{z}_2 = (0, 1, -2, -1),$$

effectively satisfying the *orthogonality conditions* with the *basis eigenvectors* $\boldsymbol{u}_{(1)}^{(a)}, \boldsymbol{u}_{(1)}^{(b)}$ of $\mathbb{E}(1)$

$$\left\langle \boldsymbol{z}_1, \boldsymbol{u}_{(1)}^{(a)} \right\rangle = 0, \qquad \left\langle \boldsymbol{z}_1, \boldsymbol{u}_{(1)}^{(b)} \right\rangle = 0, \qquad \left\langle \boldsymbol{z}_2, \boldsymbol{u}_{(1)}^{(a)} \right\rangle = 0, \qquad \left\langle \boldsymbol{z}_2, \boldsymbol{u}_{(1)}^{(b)} \right\rangle = 0.$$

Exercise 2.

The homogeneous equation associated to the given equation is $4y''(x) + 4y'(x) + y(x) = 0$, to which the algebraic equation $4\lambda^2 + 4\lambda + 1 = 0$ corresponds, having the solution $\lambda = -1/2$ with algebraic multiplicity 2.

The solution, that we denote by $y_0(x)$, of the homogeneous equation is then

$$y_0(x) = Ae^{-x/2} + Bxe^{-x/2},$$

and since the right-hand side of the given non-homogeneous equation is $3e^{-x/2}$, that is the product of a constant (polynomial of zeroth degree) times the exponential $e^{-x/2}$, we write the *particular solution* $y_p(x)$ in the same form $y_p(x) = ke^{-x/2}$. Since this $y_p(x)$ is similar to the term $Ae^{-x/2}$ of the solution of the homogeneous equation, we multiply $y_p(x)$ times x and obtain the new *particular solution* $y_p(x) = kxe^{-x/2}$, which is similar to the term $Bxe^{-x/2}$ of the solution of the homogeneous equation. We then multiply $kxe^{-x/2}$ by another factor x in such a way that the final *particular solution* $y_p(x)$ assumes the final form $y_p(x) = kx^2e^{-x/2}$ and the global solution of the given equation is the function $y(x) = y_0(x) + y_p(x)$, having no pair of similar terms. Whereas the arbitrary constants A, B of $y_0(x)$ can be obtained through the *initial conditions*, the coefficient k of $y_p(x)$ has to be obtained by imposing that $y_p(x)$ (together with its derivatives) satisfies the given non-homogeneous equation.

The derivatives of $y_p(x)$ are

$$y_p'(x) = 2kxe^{-x/2} - \frac{k}{2}\,x^2e^{-x/2} \qquad \text{and} \qquad y_p''(x) = 2ke^{-x/2} - 2kxe^{-x/2} + \frac{k}{4}\,x^2e^{-x/2},$$

that, inserted into the given equation, give the equality

$$4\left(2ke^{-x/2} - 2kxe^{-x/2} + \frac{k}{4}\,x^2e^{-x/2}\right) + 4\left(2kxe^{-x/2} - \frac{k}{2}\,x^2e^{-x/2}\right) + kx^2e^{-x/2} = 3e^{-x/2},$$

from which, after the semplifications (according to the colors)

$$8ke^{-x/2} \cancel{-8kxe^{-x/2}} \cancel{+kx^2e^{-x/2}} \cancel{+8kxe^{-x/2}} \cancel{-2kx^2e^{-x/2}} + \cancel{kx^2e^{-x/2}} = 3e^{-x/2},$$

we get $8ke^{-x/2} = 3e^{-x/2}$, that is the equality $8k = 3$ between the corresponding coefficients and then $k = 3/8$.

The solution of the given differential equation is then

$$y(x) = Ae^{-x/2} + Bxe^{-x/2} + \frac{3}{8}\,x^2e^{-x/2},$$

whose first derivative is

$$y'(x) = -\frac{A}{2}\,e^{-x/2} + Be^{-x/2} - \frac{B}{2}\,xe^{-x/2} + \frac{3}{4}\,xe^{-x/2} - \frac{3}{16}\,x^2e^{-x/2},$$

from which, by imposing the *initial conditions* $y(0) = -2, y'(0) = 2$ of the *Cauchy problem*, the system

$$\begin{cases} A & = & -2 \\ -A/2 + B & = & 2 \end{cases}$$

follows, having solution $A = -2, B = 1$. The solution of the given *Cauchy problem* is then

$$y(x) = -2e^{-x/2} + xe^{-x/2} + \frac{3}{8}x^2e^{-x/2}.$$

Exercise 3. The *Lagrangian function* $\mathcal{L}(x, y, z; \lambda)$ associated to the given optimization problem is

$$\mathcal{L}(x, y, z; \lambda) = x + 2y - 3z + \lambda(2x^2 + y^2 - z^2 + x + z + 3),$$

from which the *first order conditions*

$$\begin{cases} 1 + 4\lambda x + \lambda = 0 \\ 2 + 2\lambda y = 0 \\ -3 - 2\lambda z + \lambda = 0 \\ 2x^2 + y^2 - z^2 + x + z + 3 = 0 \end{cases}$$

follow. From the first, second, and third equation, we get

$$x = -\frac{\lambda + 1}{4\lambda}, \qquad y = -\frac{1}{\lambda}, \qquad z = \frac{\lambda - 3}{2\lambda},$$

respectively, that, inserted into the fourth equation, give

$$2\left(-\frac{\lambda+1}{4\lambda}\right)^2 + \left(-\frac{1}{\lambda}\right)^2 - \left(\frac{\lambda-3}{2\lambda}\right)^2 - \frac{\lambda+1}{4\lambda} + \frac{\lambda-3}{2\lambda} + 3 = 0 \qquad \Longrightarrow \qquad \frac{25\lambda^2 - 9}{8\lambda^2} = 0,$$

where $\lambda \neq 0$ because $\lambda = 0$ can not be a *Lagrange's multiplier*.

From $25\lambda^2 - 9 = 0$, we get $\lambda = \pm 3/5$ and then the *optimal points* $(x, y, z; \lambda)$ having coordinates

$$A = \left(-\frac{2}{3}, -\frac{5}{3}, -2; \frac{3}{5}\right) \qquad \text{and} \qquad B = \left(\frac{1}{6}, \frac{5}{3}, 3; -\frac{3}{5}\right).$$

The *bordered hessian matrix* of this optimization problem is

$$\overline{H}(x, y, z; \lambda) = \begin{pmatrix} 0 & 4x+1 & 2y & 1-2z \\ 4x+1 & 4\lambda & 0 & 0 \\ 2y & 0 & 2\lambda & 0 \\ 1-2z & 0 & 0 & -2\lambda \end{pmatrix},$$

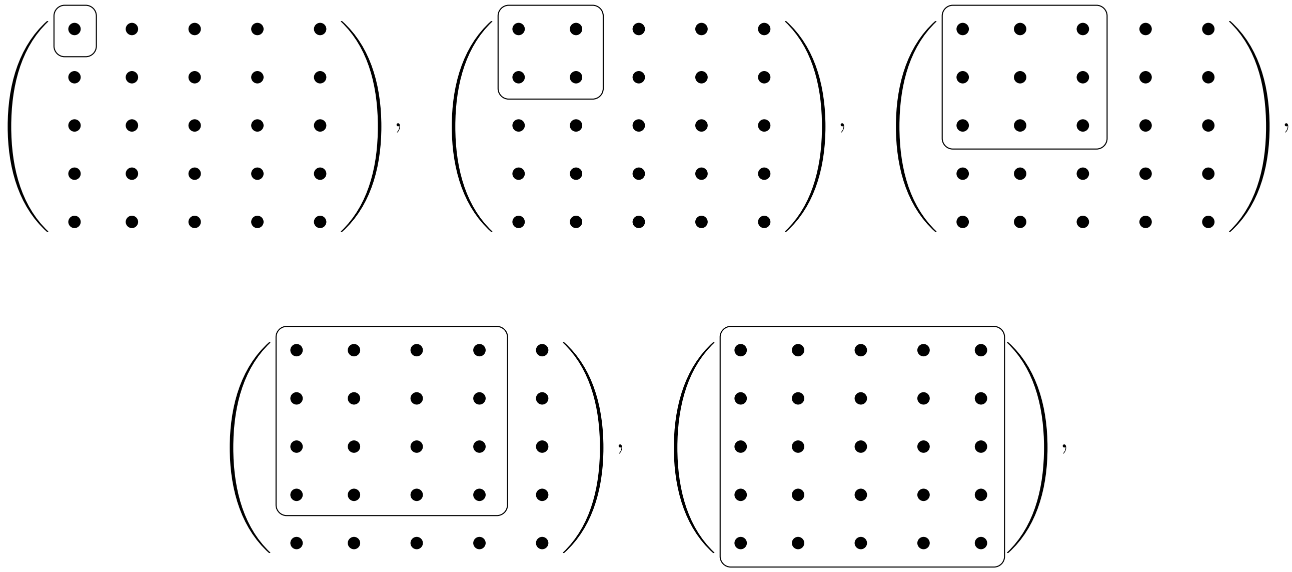
and we remind the general *second order conditions* based on the analysis of the *bordered hessian matrix*.

Given a square matrix $\overline{H}$ of order n and a positive integer number $k \leqslant n$, the *minor matrix* consisting of the first k rows and the first k columns of $\overline{H}$ is called *leading principal minor* of order k included in the matrix $\overline{H}$.

In order to fix the ideas, we consider for example a square matrix of order 5

$$\overline{H} = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix},$$

in which we highlight all *leading principal minors*, from the order 1 until the highest possible order 5



and we denote by $\mathcal{H}_k$ the *determinant* of the *leading principal minor* of order k included in the matrix $\overline{H}$.

The general *second order conditions* based on the analysis of the *bordered hessian matrix* $\overline{H}$ now read in the following way. Given the *optimization problem* consisting of optimizing a function depending on n variables subject to $p < n$ constraints, we consider the *bordered hessian matrix* $\overline{H}(P)$ corresponding to the *optimization problem*, evaluated in an *optimal point* P determined by means of the *first order conditions*. We then have that

- if it yields
$$\begin{array}{c}(-1)^{p+1}\mathcal{H}_{2p+1}(P) > 0,\\ (-1)^{p+2}\mathcal{H}_{2p+2}(P) > 0,\\ (-1)^{p+3}\mathcal{H}_{2p+3}(P) > 0,\\ \vdots \\ (-1)^{n}\mathcal{H}_{n+p}(P) > 0,\end{array} \tag{3a}$$
the point P is the *maximum point*;

- if it yields
$$\begin{array}{c}(-1)^{p}\mathcal{H}_{2p+1}(P) > 0,\\ (-1)^{p}\mathcal{H}_{2p+2}(P) > 0,\\ (-1)^{p}\mathcal{H}_{2p+3}(P) > 0,\\ \vdots \\ (-1)^{p}\mathcal{H}_{n+p}(P) > 0,\end{array} \tag{3b}$$
the point P is the *minimum point*.

It is important to point out that conditions (3) are *sufficient conditions*, only, and it is also possible that they do not hold. If conditions (3) do not hold, we have to conclude that the *nature* of the *optimal point* can not be determined by means of the *second order conditions* (3), and conditions of higher order are have to be studied.

In the exercise of the exam, we have the *bordered hessian matrices* evaluated in the two *optimal points* A, B

$$\overline{H}(A) = \begin{pmatrix} 0 & -5/3 & -10/3 & 5 \\ -5/3 & 12/5 & 0 & 0 \\ -10/3 & 0 & 6/5 & 0 \\ 5 & 0 & 0 & -6/5 \end{pmatrix} \quad \text{and} \quad \overline{H}(B) = \begin{pmatrix} 0 & 5/3 & 10/3 & -5 \\ 5/3 & -12/5 & 0 & 0 \\ 10/3 & 0 & -6/5 & 0 \\ -5 & 0 & 0 & 6/5 \end{pmatrix}.$$

Since we have $n = 3$ variables and $p = 1$ constraint, we have $2p + 1 = 3$ and $n + p = 4$, that is we have to compute the determinant of the *leading principal minors* of order 3 and of order 4 of the *bordered hessian matrices* $\overline{H}(A), \overline{H}(B)$ evaluated in the *optimal points*.

The *leading principal minors* of order 3 and of order 4 of $\overline{H}(A)$ have determinant

$$\det\begin{pmatrix} 0 & -5/3 & -10/3 \\ -5/3 & 12/5 & 0 \\ -10/3 & 0 & 6/5 \end{pmatrix} = \left[\frac{5}{3}\det\begin{pmatrix} -5/3 & 0 \\ -10/3 & 6/5 \end{pmatrix}\right] - \left[\frac{10}{3}\det\begin{pmatrix} -5/3 & 12/5 \\ -10/3 & 0 \end{pmatrix}\right] =$$

$$= \left[\left(\frac{5}{3}\right)(-2)\right] - \left[\left(\frac{10}{3}\right)(8)\right] = -30 < 0$$

and

$$\det\overline{H}(A) = \det\begin{pmatrix} 0 & -5/3 & -10/3 & 5 \\ -5/3 & 12/5 & 0 & 0 \\ -10/3 & 0 & 6/5 & 0 \\ 5 & 0 & 0 & -6/5 \end{pmatrix} =$$

$$= -5\det\begin{pmatrix} -5/3 & -10/3 & 5 \\ 12/5 & 0 & 0 \\ 0 & 6/5 & 0 \end{pmatrix} - \frac{6}{5}\det\begin{pmatrix} 0 & -5/3 & -10/3 \\ -5/3 & 12/5 & 0 \\ -10/3 & 0 & 6/5 \end{pmatrix} =$$

$$= -5\left(-\frac{6}{5}\right)\det\begin{pmatrix} -5/3 & 5 \\ 12/5 & 0 \end{pmatrix} - \frac{6}{5}\left[\frac{5}{3}\det\begin{pmatrix} -5/3 & 0 \\ -10/3 & 6/5 \end{pmatrix} - \frac{10}{3}\det\begin{pmatrix} -5/3 & 12/5 \\ -10/3 & 0 \end{pmatrix}\right] =$$

$$= \left[-5\left(-\frac{6}{5}\right)(-12)\right] - \left\{\frac{6}{5}\left[\frac{5}{3}(-2) - \frac{10}{3}(8)\right]\right\} = -72 + 36 = -36 < 0,$$

that is the *leading principal minors* $\mathcal{H}_3(A), \mathcal{H}_4(A)$ fullfil the conditions (3b)

$$-\mathcal{H}_3(A) > 0 \qquad \text{and} \qquad -\mathcal{H}_4(A) > 0,$$

from which we can conclude that the point A is the *minimum point*.

By observing that the elements of the *bordered hessian matrices* $\overline{H}(B)$ have the opposite sign with respect to the elements of the *bordered hessian matrices* $\overline{H}(A)$, we have that the determinant of the *leading principal minor* $\mathcal{H}_3(B)$ has opposite sign with respect to the determinant of the *leading principal minor* $\mathcal{H}_3(A)$, because $\mathcal{H}_3$ is a matrix of odd order, whereas the determinant of the *leading principal minor* $\mathcal{H}_4(B)$ has the same sign of the determinant of the *leading principal minor* $\mathcal{H}_4(A)$, because $\mathcal{H}_4$ is a matrix of even order.

From $\mathcal{H}_3(B) > 0$ and $\mathcal{H}_4(B) < 0$, it follows that the *leading principal minors* $\mathcal{H}_3(B), \mathcal{H}_4(B)$ fullfil the conditions (3a)

$$\mathcal{H}_3(B) > 0 \qquad \text{and} \qquad -\mathcal{H}_4(B) > 0,$$

from which we obtain that the point B is the *maximum point*.

MATHEMATICS FOR FINANCE Exam

February 2024, the 6th

Surname ______________________________ **Name** ______________________________

ID Number ________________________

Exercise 1. Given the canonical basis $\mathcal{B}_{\mathbb{R}^4} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ of the vector spaces $\mathbb{R}^4$, and the linear application $L : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ acting on the basis vectors of $\mathbb{R}^4$ according the transformation laws

$$\begin{cases} L(\boldsymbol{e}_1) = -2\boldsymbol{e}_1 + \boldsymbol{e}_3 + \boldsymbol{e}_4 \\ L(\boldsymbol{e}_2) = -2\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 - 3\boldsymbol{e}_4 \\ L(\boldsymbol{e}_3) = -2\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 \\ L(\boldsymbol{e}_4) = 3\boldsymbol{e}_2 \,, \end{cases}$$

1) write the matrix A associated to the linear application L with respect to the given basis;
2) find the subspaces *kernel* and *image* of the linear application L determining their dimension and a basis for both subspaces;
3) find the *orthogonal projection* of the vector $\boldsymbol{u} = (1, 0, 2, -1)$ on the subspace *image* of L.

Let us consider the linear application $\tilde{L} : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ defined by the transformation laws of the components

$$\tilde{L}(x_1, x_2, x_3, x_4) = (x_3, x_4, x_1 - x_4, -x_2),$$

where in the vector space $\mathbb{R}^4$ the same basis $\mathcal{B}_{\mathbb{R}^4}$ is fixed as before.

4) Write the matrix B associated to the linear application $\tilde{L}$ with respect to the given basis and determine the matrix, denoted by M, associated to the composition of linear applications $L \circ \tilde{L}$ (matrix product AB).
5) Verify whether the matrix M is diagonalizable.

If M is diagonalizable,

6) find the basis vectors with respect to which the matrix M assumes a diagonal form denoted by $\mathcal{D}$ and write the matrix C of the basis change such that $C^{-1}MC = \mathcal{D}$;
7) write the diagonal matrix $\mathcal{D}$ (without performing the matrix multiplication $C^{-1}MC$);
8) in the eigenspace of the matrix M corresponding to the eigenvalue having algebraic multiplicity 2, find an eingenvector $\boldsymbol{v}$ of M which is orthogonal to the vector $\boldsymbol{w} = (-3, 1, 4, -1)$;
9) find a basis of the subspace *orthogonal complement* of the eigenspace of the matrix M corresponding to the eigenvalue having algebraic multiplicity 2.

Exercise 2. Solve the following Cauchy problem

$$\begin{cases} y''(x) + 4y'(x) + 4y(x) = (6x - 2)e^{-2x} \\ y(0) = 1 \\ y'(0) = -1. \end{cases}$$

Exercise 3. Find the optimal points of the function

$$f(x, y, z) = 3x - 3y + 2z$$

subject to the constraint $x^2 - y^2 - z^2 + 3x + z = -11.$

Solution of the exam of the day February 2024, the 6th

Exercise 1.

1) The matrix A is

$$A = \begin{pmatrix} -2 & -2 & -2 & 0 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 1 & 0 \\ 1 & -3 & 0 & 0 \end{pmatrix},$$

obtained by writing in columns the coefficients of

$$L(\boldsymbol{e}_1) = -2\boldsymbol{e}_1 + \boldsymbol{e}_3 + \boldsymbol{e}_4\,, \qquad L(\boldsymbol{e}_2) = -2\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 - 3\boldsymbol{e}_4\,,$$
$$L(\boldsymbol{e}_3) = -2\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3\,, \qquad L(\boldsymbol{e}_4) = 3\boldsymbol{e}_2\,.$$

2) The *kernel* of L is the subspace of $\mathbb{R}^4$ containing the vectors $\boldsymbol{k} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ such that the equality $L(\boldsymbol{k}) = \mathbf{0}$ holds, that is

$$\begin{pmatrix} -2 & -2 & -2 & 0 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 1 & 0 \\ 1 & -3 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which is an algebraic linear system having *rank* 3, because

$$\det\begin{pmatrix} -2 & -2 & -2 & 0 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 1 & 0 \\ 1 & -3 & 0 & 0 \end{pmatrix} = 3\det\begin{pmatrix} -2 & -2 & -2 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix} = 3\left[\det\begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix} + 3\det\begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix}\right] = 0$$

and the *minor* of order 3

$$\mathfrak{M} = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

highlighted in the matrix A as shown

$$A = \begin{pmatrix} -2 & -2 & -2 & 0 \\ \boxed{0} & 1 & \boxed{1 \quad 3} \\ \boxed{1} & 1 & \boxed{1 \quad 0} \\ \boxed{1} & -3 & \boxed{0 \quad 0} \end{pmatrix},$$

has determinant

$$\det\mathfrak{M} = \det\begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \det\begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix} = -3 \neq 0.$$

By virtue of this *minor* $\mathfrak{M}$, we can extract the system

$$\begin{cases} x_3 + 3x_4 = -t \\ x_1 + x_3 = -t \\ x_1 = 3t \end{cases}$$

where we have given the arbitrary value $x_2 = t$ to the unknown x_2 that lays out of the *minor* $\mathfrak{M}$ highlighted in the matrix A. The *kernel* has then dimension 1 because this linear system has the $\infty^{4-3} = \infty^1$ solutions

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -4 \\ 1 \end{pmatrix} t,$$

from which we get that a basis vector of the *kernel* is the vector $\boldsymbol{k} = (3, 1, -4, 1)$, as it can be verified through

$$L(\boldsymbol{k}) = A\boldsymbol{k} = \begin{pmatrix} -2 & -2 & -2 & 0 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 1 & 0 \\ 1 & -3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The *image* of L is spanned by all those column vectors having some component contained inside the *minor* highlighted in the matrix A, that is we have the basis of the *image*

$$\mathcal{B}_{Im(L)} = \boldsymbol{w}_1 = \left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{w}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{w}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\},$$

that is the first, third, and fourth column of A, where the fourth column for $\boldsymbol{w}_3$ has been divided by 3.

3) The *orthogonal projection* of the vector $\boldsymbol{u}$ on the *image* of L is the vector, that we denote by $\boldsymbol{p}$ belonging to the *image*, such that it yields

$$\langle \boldsymbol{u} - \boldsymbol{p}\,,\, \boldsymbol{w}_1 \rangle = 0, \qquad \langle \boldsymbol{u} - \boldsymbol{p}\,,\, \boldsymbol{w}_2 \rangle = 0, \qquad \langle \boldsymbol{u} - \boldsymbol{p}\,,\, \boldsymbol{w}_3 \rangle = 0. \tag{4}$$

By expanding the vector $\boldsymbol{p} \in Im(L)$ as linear combination of the basis vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ of the *image*, that is

$$\boldsymbol{p} = \alpha \boldsymbol{w}_1 + \beta \boldsymbol{w}_2 + \gamma \boldsymbol{w}_3\,, \tag{5}$$

we have

$$\boldsymbol{u} - \boldsymbol{p} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix} - \alpha \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \beta \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \gamma \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 + 2\alpha + 2\beta \\ -\beta - \gamma \\ 2 - \alpha - \beta \\ -1 - \alpha \end{pmatrix},$$

by virtue of which the three equations (4) assume the form of the linear system

$$\begin{cases} 6\alpha + 5\beta = -1 \\ -5\alpha - 6\beta - \gamma = 0 \\ \beta + \gamma = 0. \end{cases}$$

The sum of the three equations gives the result $\alpha = -1, \beta = 1, \gamma = -1$, and then, from (5), the *orthogonal projection* $\boldsymbol{p} = (0, 0, 0, -1)$.

4) The matrix B associated to the linear application $\tilde{L}(x_1, x_2, x_3, x_4) = (x_3, x_4, x_1 - x_4, -x_2)$ is the matrix

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

because it reproduces the given transformation laws of $\tilde{L}$, that is

$$\tilde{L} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_1 - x_4 \\ -x_2 \end{pmatrix}.$$

From the matrix B, one gets the matrix M associated to the product of linear applications in the order $L\tilde{L}$

$$M = AB = \begin{pmatrix} -2 & -2 & -2 & 0 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 1 & 0 \\ 1 & -3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 0 & -2 & 0 \\ 1 & -3 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 \end{pmatrix}.$$

5,6) In order to verify whether the matrix M, which is an *endomorphism* of $\mathbb{R}^4$, is *diagonalizable*, we have to extablish whether there exists a basis of the vector space $\mathbb{R}^4$ consisting of four eigenvectors of M, that is we have to verify, in other words, whether there exist four *linearly independent* eigenvectors of M, which are *basis eigenvectors* of their corresponding eigenspaces, denoted by $\mathbb{E}(\lambda_i)$, where λ_i represents an eigenvalue of M.

Due to the expansion of the determinant according to the fourth column, the *characteristic polynomial* of M is

$$\det(M - \lambda\mathbb{I}) = \det \begin{pmatrix} -2-\lambda & 0 & -2 & 0 \\ 1 & -3-\lambda & 0 & 0 \\ 1 & 0 & 1-\lambda & 0 \\ 0 & 0 & 1 & -3-\lambda \end{pmatrix} = (-3-\lambda)\det \begin{pmatrix} -2-\lambda & 0 & -2 \\ 1 & -3-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix} =$$

$$= (-3-\lambda)(-3-\lambda)\det \begin{pmatrix} -2-\lambda & -2 \\ 1 & 1-\lambda \end{pmatrix} = \lambda(\lambda+3)^2\,(\lambda+1),$$

whose zeros are:

- the *simple*[2] eigenvalues $\lambda = 0$ and $\lambda = -1$,
- the eigenvalue $\lambda = -3$, having *algebraic multiplicity* 2.

To the *simple* eigenvalue $\lambda = 0$ we associate the linear system $(M - 0\mathbb{I})\boldsymbol{u} = \mathbf{0}$, that is

$$\begin{pmatrix} -2 & 0 & -2 & 0 \\ 1 & -3 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

having *rank* 3 by virtue of the following *minor matrix* of order 3 highlighted in M

$$M = \begin{pmatrix} -2 & 0 & -2 & 0 \\ \boxed{\begin{matrix} 1 & -3 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{matrix}} & \begin{matrix} 0 \\ 0 \\ -3 \end{matrix} \end{pmatrix},$$

from which it follows that the system has ∞^1 solutions, and the eigenspace $\mathbb{E}(0)$ has *dimension* 1.

By virtue of the highlighted *minor matrix*, we put $x_4 = t$ and solve $x_1 - 3x_2 = 0, x_1 + x_3 = 0, x_3 = 3t$, from which we get $-x_1 = x_3 = 3t, x_2 = t$ and then the first eigenvector $\boldsymbol{u}_{(0)} = (-3, -1, 3, 1)$ as basis eigenvector of the eigenspace $\mathbb{E}(0)$, satisfying effectively the equality

$$\begin{pmatrix} -2 & 0 & -2 & 0 \\ 1 & -3 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} -3 \\ -1 \\ 3 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} -3 \\ -1 \\ 3 \\ 1 \end{pmatrix}, \qquad \text{that is} \qquad M\boldsymbol{u}_{(0)} = 0\boldsymbol{u}_{(0)}.$$

[2]We remind that an eigenvalue λ of a matrix is called *simple eigenvalue* if its *algebraic multiplicity* is 1.

To the *simple* eigenvalue $\lambda = -1$, we associate the linear system $[M - (-1)\mathbb{I}]\boldsymbol{u} = \mathbf{0}$, that is

$$\begin{pmatrix} -1 & 0 & -2 & 0 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

having *rank* 3 by virtue of the following *minor matrix* of order 3 highlighted in $M + \mathbb{I}$

$$M + \mathbb{I} = \begin{pmatrix} -1 & 0 & -2 & 0 \\ 1 & \boxed{-2} & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2 \end{pmatrix},$$

from which it follows that the system has ∞^1 solutions, and the eigenspace $\mathbb{E}(-1)$ has *dimension* 1.

By virtue of the highlighted *minor matrix*, we put $x_1 = t$ and solve $-2x_2 = -t, 2x_3 = -t, x_3 - 2x_4 = 0$, from which we get $x_2 = t/2$ and then, by eliminating the fractions, the second eigenvector $\boldsymbol{u}_{(-1)} = (4, 2, -2, -1)$ as basis eigenvector of the eigenspace $\mathbb{E}(-1)$, satisfying effectively the equality

$$\begin{pmatrix} -2 & 0 & -2 & 0 \\ 1 & -3 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ -2 \\ -1 \end{pmatrix} = - \begin{pmatrix} 4 \\ 2 \\ -2 \\ -1 \end{pmatrix}, \qquad \text{that is} \qquad M\boldsymbol{u}_{(-1)} = -\boldsymbol{u}_{(-1)}.$$

To the eigenvalue $\lambda = -3$, having *algebraic multiplicity* 2, we associate the system $(M + 3\mathbb{I})\boldsymbol{u} = \mathbf{0}$, that is

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

having *rank* 2 by virtue of the following *minor matrix* of order 2 highlighted in $M + 3\mathbb{I}$

$$M + 3\mathbb{I} = \begin{pmatrix} \boxed{1} & 0 & \boxed{-2} & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ \boxed{0} & 0 & \boxed{1} & 0 \end{pmatrix},$$

from which it follows that the system has ∞^2 solutions, and the eigenspace $\mathbb{E}(-3)$ has *dimension* 2.

By virtue of the highlighted *minor matrix*, we put $x_2 = \alpha, x_4 = \beta$ and solve $x_1 - 2x_3 = 0, x_3 = 0$, from which we get $x_1 = x_3 = 0$ and then the last two eigenvectors

$$\boldsymbol{u}^{(a)}_{(-3)} = (0, 1, 0, 0) \qquad \text{and} \qquad \boldsymbol{u}^{(b)}_{(-3)} = (0, 0, 0, 1)$$

as basis eigenvectors of the eigenspace $\mathbb{E}(-3)$, satisfying effectively the equalities

$$\begin{pmatrix} -2 & 0 & -2 & 0 \\ 1 & -3 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = -3 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -2 & 0 & -2 & 0 \\ 1 & -3 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = -3 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

that is $M\boldsymbol{u}^{(a)}_{(-3)} = -3\boldsymbol{u}^{(a)}_{(-3)}$ and $M\boldsymbol{u}^{(b)}_{(-3)} = -3\boldsymbol{u}^{(b)}_{(-3)}$.

Since the set $\mathcal{B} = \{\boldsymbol{u}_{(0)}, \boldsymbol{u}_{(-1)}, \boldsymbol{u}^{(a)}_{(-3)}, \boldsymbol{u}^{(b)}_{(-3)}\}$, containing the four eigenvectors of the matrix M, is linearly independent, we conclude that the set $\mathcal{B}$ is a basis of the vector space $\mathbb{R}^4$, and the matrix M is *diagonalizable*.

The matrix C describing the basis change from the *initial basis* to the basis of the eigenvectors, with respect to which M assumes *diagonal form*, is then the one whose columns are the four eigenvectors, that is

$$C = \begin{pmatrix} -3 & 4 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 3 & -2 & 0 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}.$$

7) Since we have written the eigenvectors in the matrix C in the sequence corresponding to the eigenvalues in the order $\lambda = 0, -2, 1, 1$, respectively, it follows that the diagonal matrix $\mathcal{D}$, associated to M, is

$$\mathcal{D} = C^{-1}MC = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

8) The eigenspace associated to the eigenvalue having algebraic multiplicity 2 is $\mathbb{E}(-3)$, corresponding to the eigenvalue $\lambda = -3$, spanned by the two eigenvectors $\boldsymbol{u}^{(a)}_{(-3)}, \boldsymbol{u}^{(b)}_{(-3)}$. The vectors of this subspace have the parametric form $(x_1, x_2, x_3, x_4) = (0, \alpha, 0, \beta)$, and the general vector $\boldsymbol{v}$ of this subspace, orthogonal to the given vector $\boldsymbol{w} = (-3, 1, 4, -1)$, is the vector $\boldsymbol{v} = (0, \alpha, 0, \beta)$ such that the *scalar product* $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$ vanishes, that is the equality $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \langle\, (0, \alpha, 0, \beta)\,,\, (-3, 1, 4, -1)\,\rangle = 0$ holds, from which we get the relation $\alpha - \beta = 0$.

By choosing the particular solution $\alpha = 1, \beta = 1$, we finally obtain the particular vector $\boldsymbol{v} = (0, 1, 0, 1)$ belonging to the eigenspace $\mathbb{E}(-3)$ and orthogonal to the given vector $\boldsymbol{w} = (-3, 1, 4, -1)$.

9) The eigenspace $\mathbb{E}(-3)$ associated to the eigenvalue having algebraic multiplicity 2 is spanned by the two eigenvectors $\boldsymbol{u}^{(a)}_{(-3)}, \boldsymbol{u}^{(b)}_{(-3)}$ and its *orthogonal complement* consists of all vectors $\boldsymbol{v}^{\perp} = (y_1, y_2, y_3, y_4)$ orthogonal to every vector of $\mathbb{E}(-3)$ itself. By virtue of the *theorem of the orthogonal complement*, it is actually sufficient that the vectors $\boldsymbol{v}^{\perp} = (y_1, y_2, y_3, y_4)$ to be orthogonal to the basis eigenvectors $\boldsymbol{u}^{(a)}_{(-3)}, \boldsymbol{u}^{(b)}_{(1)}$ of $\mathbb{E}(-3)$, only.

Therefore, we impose the *orthogonality conditions*

$$\left\langle\, (y_1, y_2, y_3, y_4)\,,\, \boldsymbol{u}^{(a)}_{(-3)}\,\right\rangle = 0 \qquad \text{and} \qquad \left\langle\, (y_1, y_2, y_3, y_4)\,,\, \boldsymbol{u}^{(b)}_{(-3)}\,\right\rangle = 0,$$

which are equivalent to the linear system having *rank* 2 and 4 unknowns $y_2 = 0, y_4 = 0$.

Since this system has the ∞^2 solutions $(y_1, y_2, y_3, y_4) = (\alpha, 0, \beta, 0)$, we can conclude that the *basis vectors* of the *orthogonal complement* of the eigenspace $\mathbb{E}(-3)$ are $\boldsymbol{z}_1 = (1, 0, 0, 0)$ and $\boldsymbol{z}_2 = (0, 0, 1, 0)$, effectively satisfying the *orthogonality conditions* with the *basis eigenvectors* $\boldsymbol{u}^{(a)}_{(-3)}, \boldsymbol{u}^{(b)}_{(-3)}$ of $\mathbb{E}(-3)$

$$\left\langle \boldsymbol{z}_1, \boldsymbol{u}^{(a)}_{(-3)} \right\rangle = 0, \qquad \left\langle \boldsymbol{z}_1, \boldsymbol{u}^{(b)}_{(-3)} \right\rangle = 0, \qquad \left\langle \boldsymbol{z}_2, \boldsymbol{u}^{(a)}_{(-3)} \right\rangle = 0, \qquad \left\langle \boldsymbol{z}_2, \boldsymbol{u}^{(b)}_{(-3)} \right\rangle = 0.$$

Exercise 2.

The homogeneous equation associated to the given equation is $y''(x) + 4y'(x) + 4y(x) = 0$, to which the algebraic equation $\lambda^2 + 4\lambda + 4 = 0$ corresponds, having the solution $\lambda = -2$ with algebraic multiplicity 2.

The solution, that we denote by $y_0(x)$, of the homogeneous equation is then

$$y_0(x) = Ae^{-2x} + Bxe^{-2x},$$

and since the right-hand side of the given non-homogeneous equation is $6xe^{-2x} - 2e^{-2x}$, that is the product of a polynomial of first degree times the exponential e^{-2x}, we write the *particular solution* $y_p(x)$ in the same form

$$y_p(x) = (hx + k)e^{-2x}.$$

Since this $y_p(x)$ has similar terms to the ones of the solution of the homogeneous equation, we multiply $y_p(x)$ times x and obtain the new *particular solution*

$$y_p(x) = (hx^2 + kx)e^{-2x},$$

whose term with k is similar to the term Bxe^{-2x} of the solution of the homogeneous equation. We then multiply $(hx^2 + kx)e^{-2x}$ by another factor x in such a way that the final *particular solution* $y_p(x)$ assumes the final form

$$y_p(x) = (hx^3 + kx^2)e^{-2x}$$

and the global solution of the given equation is the function

$$y(x) = y_0(x) + y_p(x),$$

having no pair of similar terms. Whereas the arbitrary constants A, B of $y_0(x)$ can be obtained through the *initial conditions*, the coefficients h, k of $y_p(x)$ have to be obtained by imposing that $y_p(x)$ (together with its derivatives) satisfies the given non-homogeneous equation. The derivatives of $y_p(x)$ are

$$\begin{aligned} y_p'(x) &= 3hx^2e^{-2x} - 2hx^3e^{-2x} + 2kxe^{-2x} - 2kx^2e^{-2x}, \\ y_p''(x) &= 6hxe^{-2x} - 12hx^2e^{-2x} + 4hx^3e^{-2x} + 2ke^{-2x} - 8kxe^{-2x} + 4kx^2e^{-2x}, \end{aligned}$$

that, inserted into the given equation, give the equality

$$6hxe^{-2x} - 12hx^2e^{-2x} + 4hx^3e^{-2x} + 2ke^{-2x} - 8kxe^{-2x} + 4kx^2e^{-2x} +$$
$$+ 4\left(3hx^2e^{-2x} - 2hx^3e^{-2x} + 2kxe^{-2x} - 2kx^2e^{-2x}\right) + 4\left(hx^3e^{-2x} + kx^2e^{-2x}\right) = 6xe^{-2x} - 2e^{-2x},$$

from which, after the semplifications (according to the colors)

$$6hxe^{-2x} \cancel{-12hx^2e^{-2x}} \cancel{+4hx^3e^{-2x}} + 2ke^{-2x} \cancel{-8kxe^{-2x}} \cancel{+4kx^2e^{-2x}} +$$
$$\cancel{+12hx^2e^{-2x}} \cancel{-8hx^3e^{-2x}} \cancel{+8kxe^{-2x}} \cancel{-8kx^2e^{-2x}} \cancel{+4hx^3e^{-2x}} \cancel{+4kx^2e^{-2x}} = 6xe^{-2x} - 2e^{-2x},$$

we get

$$6hxe^{-2x} + 2ke^{-2x} = 6xe^{-2x} - 2e^{-2x},$$

that is the equalities $6h = 6, 2k = -2$ between the corresponding coefficients and then $h = 1, k = -1$.

The solution of the given differential equation is then

$$y(x) = Ae^{-2x} + Bxe^{-2x} + x^3e^{-2x} - x^2e^{-2x},$$

whose first derivative is

$$y'(x) = -2Ae^{-2x} + Be^{-2x} - 2Bxe^{-2x} + 3x^2e^{-2x} - 2x^3e^{-2x} - 2xe^{-2x} + 2x^2e^{-2x},$$

from which, by imposing the *initial conditions* $y(0) = 1, y'(0) = -1$ of the *Cauchy problem*, the system

$$\begin{cases} A &= 1 \\ -2A + B &= -1 \end{cases}$$

follows, having solution $A = 1, B = 1$. The solution of the given *Cauchy problem* is then

$$y(x) = e^{-2x} + xe^{-2x} + x^3e^{-2x} - x^2e^{-2x}.$$

Exercise 3. The *Lagrangian function* $\mathcal{L}(x, y, z; \lambda)$ associated to the given optimization problem is

$$\mathcal{L}(x, y, z; \lambda) = 3x - 3y + 2z + \lambda(x^2 - y^2 - z^2 + 3x + z + 11),$$

from which the *first order conditions*

$$\begin{cases} 3 + 2\lambda x + 3\lambda = 0 \\ -3 - 2\lambda y = 0 \\ 2 - 2\lambda z + \lambda = 0 \\ x^2 - y^2 - z^2 + 3x + z + 3 = 0 \end{cases}$$

follow. From the first, second, and third equation, we get

$$x = -\frac{3\lambda + 3}{2\lambda}, \qquad y = -\frac{3}{2\lambda}, \qquad z = \frac{\lambda + 2}{2\lambda},$$

respectively, that, inserted into the fourth equation, give

$$\left(-\frac{3\lambda+3}{2\lambda}\right)^2 - \left(-\frac{3}{2\lambda}\right)^2 - \left(\frac{\lambda+2}{2\lambda}\right)^2 - \frac{9\lambda+9}{2\lambda} + \frac{\lambda+2}{2\lambda} + 11 = 0 \qquad \Longrightarrow \qquad \frac{36\lambda^2 - 4}{4\lambda^2} = 0,$$

where $\lambda \neq 0$ because $\lambda = 0$ can not be a *Lagrange's multiplier*. From $36\lambda^2 - 4 = 0$, we get $\lambda = \pm 1/3$ and then the *optimal points* $(x, y, z; \lambda)$ having coordinates

$$A = \left(-6, -\frac{9}{2}, \frac{7}{2}; \frac{1}{3}\right) \qquad \text{and} \qquad B = \left(3, \frac{9}{2}, -\frac{5}{2}; -\frac{1}{3}\right).$$

The *bordered hessian matrix* of this optimization problem is

$$\overline{H}(x, y, z; \lambda) = \begin{pmatrix} 0 & 2x+3 & -2y & 1-2z \\ 2x+3 & 2\lambda & 0 & 0 \\ -2y & 0 & -2\lambda & 0 \\ 1-2z & 0 & 0 & -2\lambda \end{pmatrix},$$

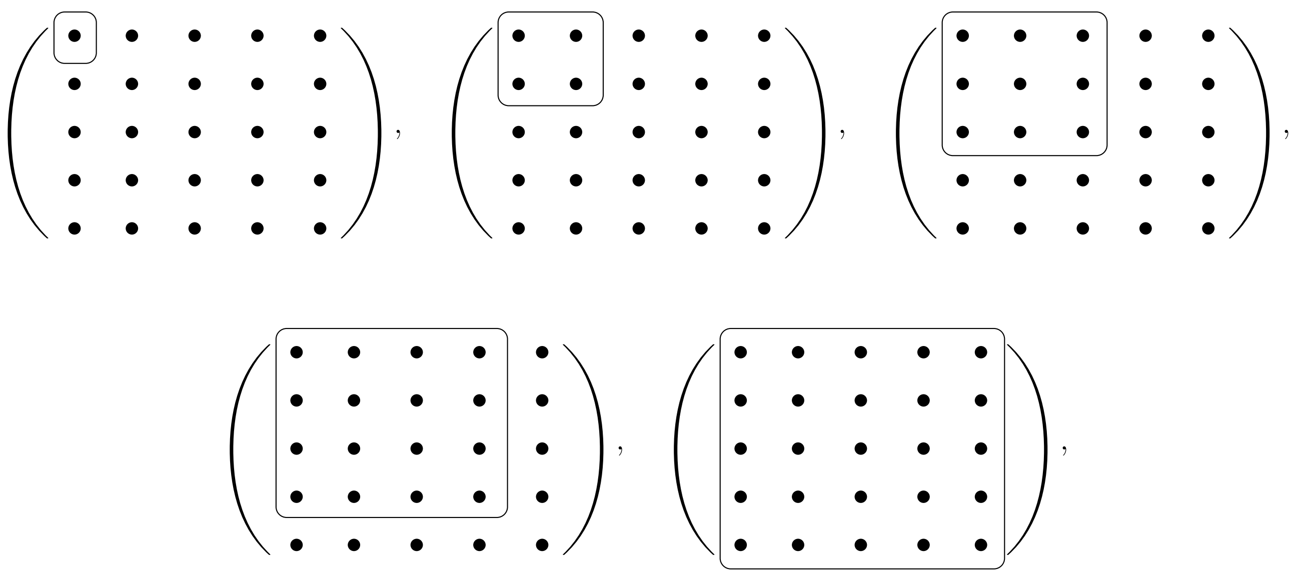
and we remind the general *second order conditions* based on the analysis of the *bordered hessian matrix*.

Given a square matrix $\overline{H}$ of order n and a positive integer number $k \leqslant n$, the *minor matrix* consisting of the first k rows and the first k columns of $\overline{H}$ is called *leading principal minor* of order k included in the matrix $\overline{H}$.

In order to fix the ideas, we consider for example a square matrix of order 5

$$\overline{H} = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix},$$

in which we highlight all *leading principal minors*, from the order 1 until the highest possible order 5



and we denote by $\mathcal{H}_k$ the *determinant* of the *leading principal minor* of order k included in the matrix $\overline{H}$.

The general *second order conditions* based on the analysis of the *bordered hessian matrix* $\overline{H}$ now read in the following way. Given the *optimization problem* consisting of optimizing a function depending on n variables subject to $p < n$ constraints, we consider the *bordered hessian matrix* $\overline{H}(P)$ corresponding to the *optimization problem*, evaluated in an *optimal point* P determined by means of the *first order conditions*. We then have that

- if it yields

$$\begin{aligned}
&(-1)^{p+1}\mathcal{H}_{2p+1}(P) > 0,\\
&(-1)^{p+2}\mathcal{H}_{2p+2}(P) > 0,\\
&(-1)^{p+3}\mathcal{H}_{2p+3}(P) > 0,\\
&\quad\vdots\\
&(-1)^{n}\mathcal{H}_{n+p}(P) > 0,
\end{aligned} \tag{6a}$$

  the point P is the *maximum point*;

- if it yields

$$\begin{aligned}
&(-1)^{p}\mathcal{H}_{2p+1}(P) > 0,\\
&(-1)^{p}\mathcal{H}_{2p+2}(P) > 0,\\
&(-1)^{p}\mathcal{H}_{2p+3}(P) > 0,\\
&\quad\vdots\\
&(-1)^{p}\mathcal{H}_{n+p}(P) > 0,
\end{aligned} \tag{6b}$$

  the point P is the *minimum point*.

It is important to point out that conditions (6) are *sufficient conditions*, only, and it is also possible that they do not hold. If conditions (6) do not hold, we have to conclude that the *nature* of the *optimal point* can not be determined by means of the *second order conditions* (6), and conditions of higher order are have to be studied.

In the exercise of the exam, we have the *bordered hessian matrices* evaluated in the two *optimal points* A, B

$$\overline{H}(A) = \begin{pmatrix} 0 & -9 & 9 & -6 \\ -9 & 2/3 & 0 & 0 \\ 9 & 0 & -2/3 & 0 \\ -6 & 0 & 0 & -2/3 \end{pmatrix} \quad \text{and} \quad \overline{H}(B) = \begin{pmatrix} 0 & 9 & -9 & 6 \\ 9 & -2/3 & 0 & 0 \\ -9 & 0 & 2/3 & 0 \\ 6 & 0 & 0 & 2/3 \end{pmatrix}.$$

Since we have $n = 3$ variables and $p = 1$ constraint, we have $2p + 1 = 3$ and $n + p = 4$, that is we have to compute the determinant of the *leading principal minors* of order 3 and of order 4 of the *bordered hessian matrices* $\overline{H}(A), \overline{H}(B)$ evaluated in the *optimal points*.

The *leading principal minors* of order 3 and of order 4 of $\overline{H}(A)$ have determinant

$$\det \begin{pmatrix} 0 & -9 & 9 \\ -9 & 2/3 & 0 \\ 9 & 0 & -2/3 \end{pmatrix} = 0$$

and

$$\det \overline{H}(A) = 16 > 0,$$

that is the *leading principal minors* $\mathcal{H}_3(A), \mathcal{H}_4(A)$ fullfil neither conditions (6a), nor conditions (6b), from which we can conclude that the *nature* of the *optimal point* A can not be determined by means of the *second order conditions* at disposal. By observing that the elements of the *bordered hessian matrices* $\overline{H}(B)$ have the opposite sign with respect to the elements of the *bordered hessian matrices* $\overline{H}(A)$, we conclude that not even the nature of the *optimal point* B can be studied by means of the *second order conditions* at disposal.

MATHEMATICS FOR FINANCE

April 2024, the 15th

Surname ______________________ **Name** ______________________

ID Number ______________________

Exercise 1. Given the canonical basis $\mathcal{B}_{\mathbb{R}^4} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ of the vector spaces $\mathbb{R}^4$, and the linear application $L : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ acting on the basis vectors of $\mathbb{R}^4$ according the transformation laws

$$\begin{cases} L(\boldsymbol{e}_1) = \boldsymbol{e}_1 + 7\boldsymbol{e}_2 - \boldsymbol{e}_3 - 2\boldsymbol{e}_4 \\ L(\boldsymbol{e}_2) = -3\boldsymbol{e}_2 - 6\boldsymbol{e}_4 \\ L(\boldsymbol{e}_3) = \boldsymbol{e}_2 + 2\boldsymbol{e}_4 \\ L(\boldsymbol{e}_4) = -\boldsymbol{e}_1 - 4\boldsymbol{e}_2 - 4\boldsymbol{e}_4 \,, \end{cases}$$

1) write the matrix A associated to the linear application L with respect to the given basis;

2) find the subspaces *kernel* and *image* of the linear application L determining their dimension and a basis for both subspaces;

3) find the *orthogonal projection* of the vector $\boldsymbol{u} = (5, 3, -12, 7)$ on the subspace *image* of L.

Let us consider the linear application $\tilde{L} : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ defined by the transformation laws of the components

$$\tilde{L}(x_1, x_2, x_3, x_4) = (x_3, -x_1 + x_4, x_2 + x_4, x_1 + x_3),$$

where in the vector space $\mathbb{R}^4$ the same basis $\mathcal{B}_{\mathbb{R}^4}$ is fixed as before.

4) Write the matrix B associated to the linear application $\tilde{L}$ with respect to the given basis and determine the matrix, denoted by M, associated to the composition of linear applications $L \circ \tilde{L}$ (matrix product AB).

5) Verify whether the matrix M is diagonalizable.

If M is diagonalizable,

6) find the basis vectors with respect to which the matrix M assumes a diagonal form denoted by $\mathcal{D}$ and write the matrix C of the basis change such that $C^{-1}MC = \mathcal{D}$;

7) write the diagonal matrix $\mathcal{D}$ (without performing the matrix multiplication $C^{-1}MC$);

8) in the eigenspace of the matrix M corresponding to the eigenvalue having algebraic multiplicity 2, find an eingenvector $\boldsymbol{v}$ of M which is orthogonal to the vector $\boldsymbol{w} = (-1, -2, 5, 3)$;

9) find a basis of the subspace *orthogonal complement* of the eigenspace of the matrix M corresponding to the eigenvalue having algebraic multiplicity 2.

Exercise 2. Solve the following Cauchy problem

$$\begin{cases} y''(x) + 6y'(x) + 9y(x) = (-2 + 6x)\, e^{-3x} \\ y(0) = -1 \\ y'(0) = 1 \end{cases}$$

Exercise 3. Find the optimal points of the function

$$f(x, y, z) = x - y + z \quad \text{subject to the constraint} \quad 2x^2 + y^2 - xy - z^2 + z = 2/7.$$

HINT.: from the two equations $\partial\mathcal{L}/\partial x = 0$ and $\partial\mathcal{L}/\partial y = 0$, you should obtain x, y in terms of λ.

Solution of the exam of the day April 2024, the 15th

Exercise 1.

1) The matrix A is

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 7 & -3 & 1 & -4 \\ -1 & 0 & 0 & 0 \\ -2 & -6 & 2 & -4 \end{pmatrix},$$

obtained by writing in the i-th column the coefficients of the result of $L(\boldsymbol{e}_i)$

$$L(\boldsymbol{e}_1) = \boldsymbol{e}_1 + 7\boldsymbol{e}_2 - \boldsymbol{e}_3 - 2\boldsymbol{e}_4\,, \qquad L(\boldsymbol{e}_2) = -\,3\boldsymbol{e}_2 - 6\boldsymbol{e}_4\,,$$
$$L(\boldsymbol{e}_3) = \boldsymbol{e}_2 + 2\boldsymbol{e}_4\,, \qquad L(\boldsymbol{e}_4) = -\,\boldsymbol{e}_1 - 4\boldsymbol{e}_2 - 4\boldsymbol{e}_4\,.$$

2) The *kernel* of L is the subspace of $\mathbb{R}^4$ containing the vectors $\boldsymbol{k} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ such that the equality $L(\boldsymbol{k}) = \boldsymbol{0}$ holds, that is

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 7 & -3 & 1 & -4 \\ -1 & 0 & 0 & 0 \\ -2 & -6 & 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which is an algebraic linear system having *rank* 3, because

$$\det \begin{pmatrix} 1 & 0 & 0 & -1 \\ 7 & -3 & 1 & -4 \\ -1 & 0 & 0 & 0 \\ -2 & -6 & 2 & -4 \end{pmatrix} = -1 \det \begin{pmatrix} 0 & 0 & -1 \\ -3 & 1 & -4 \\ -6 & 2 & -4 \end{pmatrix} = (-1)(-1)\det \begin{pmatrix} -3 & 1 \\ -6 & 2 \end{pmatrix} = 0$$

and the *minor* of order 3

$$\mathfrak{M} = \begin{pmatrix} 7 & 1 & -4 \\ -1 & 0 & 0 \\ -2 & 2 & -4 \end{pmatrix},$$

highlighted in the matrix A as shown

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ \boxed{7} & -3 & \boxed{1 \quad -4} \\ \boxed{-1} & 0 & \boxed{0 \quad 0} \\ \boxed{-2} & -6 & \boxed{2 \quad -4} \end{pmatrix},$$

has determinant

$$\det \mathfrak{M} = \det \begin{pmatrix} 7 & 1 & -4 \\ -1 & 0 & 0 \\ -2 & 2 & -4 \end{pmatrix} = \det \begin{pmatrix} 1 & -4 \\ 2 & -4 \end{pmatrix} = 4 \neq 0.$$

By virtue of this *minor* $\mathfrak{M}$, we can extract the system

$$\begin{cases} 7x_1 + x_3 - 4x_4 = 3t \\ -\,x_1 = 0 \\ -2x_1 + 2x_3 - 4x_4 = 6t \end{cases}$$

where we have given the arbitrary value $x_2 = t$ to the unknown x_2 that lays out of the *minor* $\mathfrak{M}$ highlighted in the matrix A. The *kernel* has then dimension 1 because this linear system has the $\infty^{4-3} = \infty^1$ solutions

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 0 \end{pmatrix} t,$$

from which we get that a basis vector of the *kernel* is the vector $\boldsymbol{k} = (0, 1, 3, 0)$, as it can be verified through

$$L(\boldsymbol{k}) = A\boldsymbol{k} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 7 & -3 & 1 & -4 \\ -1 & 0 & 0 & 0 \\ -2 & -6 & 2 & -4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The *image* of L is spanned by all those column vectors having some component contained inside the *minor* highlighted in the matrix A, that is we have the basis of the *image*

$$\mathcal{B}_{Im(L)} = \boldsymbol{w}_1 = \left\{ \begin{pmatrix} 1 \\ 7 \\ -1 \\ -2 \end{pmatrix}, \quad \boldsymbol{w}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \quad \boldsymbol{w}_3 = \begin{pmatrix} 1 \\ 4 \\ 0 \\ 4 \end{pmatrix} \right\},$$

that is the first, third, and fourth column of A, where the fourth column has been taken with the opposite sign.

3) The *orthogonal projection* of the vector $\boldsymbol{u}$ on the *image* of L is the vector, that we denote by $\boldsymbol{p}$ belonging to the *image*, such that it yields

$$\langle \boldsymbol{u} - \boldsymbol{p}\,,\, \boldsymbol{w}_1 \rangle = 0, \qquad \langle \boldsymbol{u} - \boldsymbol{p}\,,\, \boldsymbol{w}_2 \rangle = 0, \qquad \langle \boldsymbol{u} - \boldsymbol{p}\,,\, \boldsymbol{w}_3 \rangle = 0. \tag{7}$$

By expanding the vector $\boldsymbol{p} \in Im(L)$ as linear combination of the basis vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ of the *image*, that is

$$\boldsymbol{p} = \alpha \boldsymbol{w}_1 + \beta \boldsymbol{w}_2 + \gamma \boldsymbol{w}_3\,,$$

we have

$$\boldsymbol{u} - \boldsymbol{p} = \begin{pmatrix} 5 \\ 3 \\ -12 \\ 7 \end{pmatrix} - \alpha \begin{pmatrix} 1 \\ 7 \\ -1 \\ -2 \end{pmatrix} - \beta \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} - \gamma \begin{pmatrix} 1 \\ 4 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 - \alpha - \gamma \\ 3 - 7\alpha - \beta - 4\gamma \\ -12 + \alpha \\ 7 + 2\alpha - 2\beta - 4\gamma \end{pmatrix},$$

by virtue of which the three equations (7) assume the form of the linear system

$$\begin{cases} 55\alpha + 3\beta + 21\gamma = 24 \\ 3\alpha + 5\beta + 12\gamma = 17 \\ 7\alpha + 4\beta + 11\gamma = 15, \end{cases}$$

in which the third equation has been divided by 3. By subtracting the third equation multiplied by 3 from the first equation multiplied by 4, we get the equation $199\alpha + 51\gamma = 51$, whereas by subtracting the second equation multiplied by 4 from the third equation multiplied by 5, we get the equation $23\alpha + 7\gamma = 7$.

By applying Cramer's rule to the unknown α of the system

$$\begin{cases} 199\alpha + 51\gamma = 51 \\ 23\alpha + 7\gamma = 7, \end{cases}$$

we get $\alpha = 0$ and then $\beta = 1, \gamma = 1$, from which the *orthogonal projection* $\boldsymbol{p} = (1, 5, 0, 6)$ follows.

4) The matrix B associated to the linear application $\tilde{L}(x_1, x_2, x_3, x_4) = (x_3, -x_1 + x_4, x_2 + x_4, x_1 + x_3)$ is the matrix

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

because it reproduces the given transformation laws of $\tilde{L}$, that is

$$\tilde{L}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}0&0&1&0\\-1&0&0&1\\0&1&0&1\\1&0&1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}x_3\\-x_1+x_4\\x_2+x_4\\x_1+x_3\end{pmatrix}.$$

From the matrix B, one gets the matrix M associated to the product of linear applications in the order $L\tilde{L}$

$$M=AB=\begin{pmatrix}1&0&0&-1\\7&-3&1&-4\\-1&0&0&0\\-2&-6&2&-4\end{pmatrix}\begin{pmatrix}0&0&1&0\\-1&0&0&1\\0&1&0&1\\1&0&1&0\end{pmatrix}=\begin{pmatrix}-1&0&0&0\\-1&1&3&-2\\0&0&-1&0\\2&2&-6&-4\end{pmatrix}.$$

5,6) In order to verify whether the matrix M, which is an *endomorphism* of $\mathbb{R}^4$, is *diagonalizable*, we have to extablish whether there exists a basis of the vector space $\mathbb{R}^4$ consisting of four eigenvectors of M, that is we have to verify, in other words, whether there exist four *linearly independent* eigenvectors of M, which are *basis eigenvectors* of their corresponding eigenspaces, denoted by $\mathbb{E}(\lambda_i)$, where λ_i represents an eigenvalue of M.

Due to the expansion of the determinant according to the first row, the *characteristic polynomial* of M is

$$\det(M-\lambda\mathbb{I})=\det\begin{pmatrix}-1-\lambda&0&0&0\\-1&1-\lambda&3&-2\\0&0&-1-\lambda&0\\2&2&-6&-4-\lambda\end{pmatrix}=(-1-\lambda)\det\begin{pmatrix}1-\lambda&3&-2\\0&-1-\lambda&0\\2&-6&-4-\lambda\end{pmatrix}=$$

$$=(-1-\lambda)(-1-\lambda)\det\begin{pmatrix}1-\lambda&-2\\2&-4-\lambda\end{pmatrix}=\lambda(\lambda+3)(\lambda+1)^2,$$

whose zeros are:

- the *simple*[3] eigenvalues $\lambda=0$ and $\lambda=-3$,
- the eigenvalue $\lambda=-1$, having *algebraic multiplicity* 2.

To the *simple* eigenvalue $\lambda=0$ we associate the linear system $(M-0\mathbb{I})\boldsymbol{u}=\mathbf{0}$, that is

$$\begin{pmatrix}-1&0&0&0\\-1&1&3&-2\\0&0&-1&0\\2&2&-6&-4\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},$$

having *rank* 3 by virtue of the following *minor matrix* of order 3 highlighted in M

$$M=\begin{pmatrix}-1&0&0&0\\\boxed{\begin{matrix}-1&1&3\\0&0&-1\\2&2&-6\end{matrix}}&\begin{matrix}-2\\0\\-4\end{matrix}\end{pmatrix},$$

from which it follows that the system has ∞^1 solutions, and the eigenspace $\mathbb{E}(0)$ has *dimension* 1.

[3] We remind that an eigenvalue λ of a matrix is called *simple eigenvalue* if its *algebraic multiplicity* is 1.

By virtue of the highlighted *minor matrix*, we put $x_4 = t$ and solve the system

$$\begin{cases} -x_1 + x_2 + 3x_3 = 2t \\ -x_3 = 0 \\ 2x_1 + 2x_2 - 6x_3 = 4t, \end{cases}$$

from which we get $x_1 = x_3 = 0, x_2 = 2t$ and then the first eigenvector $\boldsymbol{u}_{(0)} = (0, 2, 0, 1)$ as basis eigenvector of the eigenspace $\mathbb{E}(0)$, satisfying effectively the equality

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 3 & -2 \\ 0 & 0 & -1 & 0 \\ 2 & 2 & -6 & -4 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \qquad \text{that is} \qquad M\boldsymbol{u}_{(0)} = 0\boldsymbol{u}_{(0)}.$$

To the *simple* eigenvalue $\lambda = -3$, we associate the linear system $[M - (-3)\mathbb{I}]\boldsymbol{u} = \mathbf{0}$, that is

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 4 & 3 & -2 \\ 0 & 0 & 2 & 0 \\ 2 & 2 & -6 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

having *rank* 3 by virtue of the following *minor matrix* of order 3 highlighted in $M + 3\mathbb{I}$

$$M + 3\mathbb{I} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ \boxed{\begin{matrix} -1 & 4 & 3 \\ 0 & 0 & 2 \\ 2 & 2 & -6 \end{matrix}} & \begin{matrix} -2 \\ 0 \\ -1 \end{matrix} \end{pmatrix},$$

from which it follows that the system has ∞^1 solutions, and the eigenspace $\mathbb{E}(-3)$ has *dimension* 1.

By virtue of the highlighted *minor matrix*, we put $x_4 = t$ and solve the system

$$\begin{cases} -x_1 + 4x_2 + 3x_3 = 2t \\ 2x_3 = 0 \\ 2x_1 + 2x_2 - 6x_3 = t \end{cases}$$

from which we get $x_2 = t/2$ and then, by eliminating the fractions, the second eigenvector $\boldsymbol{u}_{(-3)} = (0, 1, 0, 2)$ as basis eigenvector of the eigenspace $\mathbb{E}(-3)$, satisfying effectively the equality

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 3 & -2 \\ 0 & 0 & -1 & 0 \\ 2 & 2 & -6 & -4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} = -3 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \qquad \text{that is} \qquad M\boldsymbol{u}_{(-3)} = -3\boldsymbol{u}_{(-3)}.$$

To the eigenvalue $\lambda = -1$, having *algebraic multiplicity* 2, we associate the system $(M + \mathbb{I})\boldsymbol{u} = \mathbf{0}$, that is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 2 & 3 & -2 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & -6 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

having *rank* 2 by virtue of the following *minor matrix* of order 2 highlighted in $M + \mathbb{I}$

$$M + \mathbb{I} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \boxed{-1 \quad 2} & & 3 & -2 \\ 0 & 0 & 0 & 0 \\ \boxed{2 \quad 2} & & -6 & -3 \end{pmatrix},$$

from which it follows that the system has ∞^2 solutions, and the eigenspace $\mathbb{E}(-1)$ has *dimension* 2.

By virtue of the highlighted *minor matrix*, we put $x_3 = \alpha, x_4 = \beta$ and solve the system

$$\begin{cases} -x_1 + 2x_2 = -3\alpha + 2\beta \\ 2x_1 + 2x_2 = 6\alpha + 3\beta \end{cases}$$

from which we get $x_1 = 3\alpha + \beta/3, x_2 = 7\beta/6$ and then the last two eigenvectors

$$\boldsymbol{u}^{(a)}_{(-1)} = (3, 0, 1, 0) \qquad \text{and} \qquad \boldsymbol{u}^{(b)}_{(-1)} = (2, 7, 0, 6)$$

as basis eigenvectors of the eigenspace $\mathbb{E}(-1)$, satisfying effectively the equalities

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 3 & -2 \\ 0 & 0 & -1 & 0 \\ 2 & 2 & -6 & -4 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} = - \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 3 & -2 \\ 0 & 0 & -1 & 0 \\ 2 & 2 & -6 & -4 \end{pmatrix} \begin{pmatrix} 2 \\ 7 \\ 0 \\ 6 \end{pmatrix} = - \begin{pmatrix} 2 \\ 7 \\ 0 \\ 6 \end{pmatrix},$$

that is $M\boldsymbol{u}^{(a)}_{(-1)} = -\boldsymbol{u}^{(a)}_{(-1)}$ and $M\boldsymbol{u}^{(b)}_{(-1)} = -\boldsymbol{u}^{(b)}_{(-1)}$.

Since the set $\mathcal{B} = \{\boldsymbol{u}_{(0)}, \boldsymbol{u}_{(-3)}, \boldsymbol{u}^{(a)}_{(-1)}, \boldsymbol{u}^{(b)}_{(-1)}\}$, containing the four eigenvectors of the matrix M, is linearly independent, we conclude that the set $\mathcal{B}$ is a basis of the vector space $\mathbb{R}^4$, and the matrix M is *diagonalizable*.

The matrix C describing the basis change from the *initial basis* to the basis of the eigenvectors, with respect to which M assumes *diagonal form*, is then the one whose columns are the four eigenvectors, that is

$$C = \begin{pmatrix} 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & -6 \end{pmatrix}.$$

7) Since we have written the eigenvectors in the matrix C in the sequence corresponding to the eigenvalues in the order $\lambda = 0, -3, -1, -1$, respectively, it follows that the diagonal matrix $\mathcal{D}$, associated to M, is

$$\mathcal{D} = C^{-1}MC = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

8) The eigenspace associated to the eigenvalue having algebraic multiplicity 2 is $\mathbb{E}(-1)$, corresponding to the eigenvalue $\lambda = -1$, spanned by the two eigenvectors $\boldsymbol{u}^{(a)}_{(-1)}, \boldsymbol{u}^{(b)}_{(-1)}$. The vectors of this subspace have the parametric form $(x_1, x_2, x_3, x_4) = (3\alpha + 2\beta, 7\beta, \alpha, 6\beta)$, and the general vector $\boldsymbol{v}$ of this subspace, orthogonal to the given vector $\boldsymbol{w} = (-1, -2, 5, 3)$, is the vector $\boldsymbol{v} = (3\alpha + 2\beta, 7\beta, \alpha, 6\beta)$ such that the *scalar product* $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$ vanishes, that is the equality $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \langle\, (3\alpha + 2\beta, 7\beta, \alpha, 6\beta)\, , \, (-3, 1, 4, -1)\, \rangle = 0$ holds, from which we get the relation $\alpha + \beta = 0$. By choosing the particular solution $\alpha = 1, \beta = -1$, we finally obtain the particular vector $\boldsymbol{v} = (1, -7, 1, -6)$ belonging to the eigenspace $\mathbb{E}(-1)$ and orthogonal to the given vector $\boldsymbol{w} = (-1, -2, 5, 3)$.

9) The eigenspace $\mathbb{E}(-1)$ associated to the eigenvalue having algebraic multiplicity 2 is spanned by the two eigenvectors $\boldsymbol{u}^{(a)}_{(-1)}, \boldsymbol{u}^{(b)}_{(-1)}$ and its *orthogonal complement* consists of all vectors $\boldsymbol{v}^{\perp} = (y_1, y_2, y_3, y_4)$ orthogonal

to every vector of $\mathbb{E}(-1)$ itself. By virtue of the *theorem of the orthogonal complement*, it is actually sufficient that the vectors $\boldsymbol{v}^{\perp} = (y_1, y_2, y_3, y_4)$ to be orthogonal to the basis eigenvectors $\boldsymbol{u}^{(a)}_{(-1)}, \boldsymbol{u}^{(1)}_{(1)}$ of $\mathbb{E}(-1)$, only.

Therefore, we impose the *orthogonality conditions*

$$\Big\langle\, (y_1, y_2, y_3, y_4)\,,\ \boldsymbol{u}^{(a)}_{(-1)} \Big\rangle = 0 \qquad \text{and} \qquad \Big\langle\, (y_1, y_2, y_3, y_4)\,,\ \boldsymbol{u}^{(b)}_{(-1)} \Big\rangle = 0,$$

which are equivalent to the linear system having *rank* 2 and 4 unknowns $y_3 = -3y_1, 6y_4 = -2y_1 - 7y_2$.

Since this system has the ∞^2 solutions $(y_1, y_2, y_3, y_4) = (3\alpha, 6\beta, -9\alpha, -\alpha - 7\beta)$, we can conclude that the *basis vectors* of the *orthogonal complement* of the eigenspace $\mathbb{E}(-1)$ are

$$\boldsymbol{z}_1 = (3, 0, -9, -1) \qquad \text{and} \qquad \boldsymbol{z}_2 = (0, 6, 0, -7),$$

effectively satisfying the *orthogonality conditions* with the *basis eigenvectors* $\boldsymbol{u}^{(a)}_{(-1)}, \boldsymbol{u}^{(b)}_{(-1)}$ of $\mathbb{E}(-1)$

$$\Big\langle \boldsymbol{z}_1, \boldsymbol{u}^{(a)}_{(-1)} \Big\rangle = 0, \qquad \Big\langle \boldsymbol{z}_1, \boldsymbol{u}^{(b)}_{(-1)} \Big\rangle = 0, \qquad \Big\langle \boldsymbol{z}_2, \boldsymbol{u}^{(a)}_{(-1)} \Big\rangle = 0, \qquad \Big\langle \boldsymbol{z}_2, \boldsymbol{u}^{(b)}_{(-1)} \Big\rangle = 0.$$

Exercise 2.

The homogeneous equation associated to the given equation is $y''(x) + 6y'(x) + 9y(x) = 0$, to which the algebraic equation $\lambda^2 + 6\lambda + 9 = 0$ corresponds, having the solution $\lambda = -3$ with algebraic multiplicity 2.

The solution, that we denote by $y_0(x)$, of the homogeneous equation is then

$$y_0(x) = Ae^{-3x} + Bxe^{-3x},$$

and since the right-hand side of the given non-homogeneous equation is $6xe^{-3x} - 2e^{-3x}$, that is the product of a polynomial of first degree times the exponential e^{-3x}, we write the *particular solution* $y_p(x)$ in the same form

$$y_p(x) = (hx + k)e^{-3x}.$$

Since this $y_p(x)$ has similar terms to the ones of the solution of the homogeneous equation, we multiply $y_p(x)$ times x and obtain the new *particular solution*

$$y_p(x) = (hx^2 + kx)e^{-3x},$$

whose term with k is similar to the term Bxe^{-3x} of the solution of the homogeneous equation. We then multiply $(hx^2 + kx)e^{-3x}$ by another factor x in such a way that the final *particular solution* $y_p(x)$ assumes the final form

$$y_p(x) = (hx^3 + kx^2)e^{-3x}$$

and the global solution of the given equation is the function

$$y(x) = y_0(x) + y_p(x),$$

having no pair of similar terms. Whereas the arbitrary constants A, B of $y_0(x)$ can be obtained through the *initial conditions*, the coefficients h, k of $y_p(x)$ have to be obtained by imposing that $y_p(x)$ (together with its derivatives) satisfies the given non-homogeneous equation. The derivatives of $y_p(x)$ are

$$\begin{aligned} y_p'(x) &= 3hx^2e^{-3x} - 3hx^3e^{-3x} + 2kxe^{-3x} - 3kx^2e^{-3x}, \\ y_p''(x) &= 6hxe^{-3x} - 18hx^2e^{-3x} + 9hx^3e^{-3x} + 2ke^{-3x} - 12kxe^{-3x} + 9kx^2e^{-3x}, \end{aligned}$$

that, inserted into the given equation, give the equality

$$\begin{gathered} 6hxe^{-3x} - 18hx^2e^{-3x} + 9hx^3e^{-3x} + 2ke^{-3x} - 12kxe^{-3x} + 9kx^2e^{-3x} + \\ + 6\,(3hx^2e^{-3x} - 3hx^3e^{-3x} + 2kxe^{-3x} - 3kx^2e^{-3x}) + 9\,(hx^3e^{-3x} + kx^2e^{-3x}) = 6xe^{-3x} - 2e^{-3x}, \end{gathered}$$

from which, after the semplifications (according to the colors)

$$6hxe^{-3x} \cancel{-18hx^2e^{-3x}} \cancel{+9hx^3e^{-3x}} + 2ke^{-3x} \cancel{-12kxe^{-3x}} \cancel{+9kx^2e^{-3x}} +$$
$$\cancel{+18hx^2e^{-3x}} \cancel{-18hx^3e^{-3x}} \cancel{+12kxe^{-3x}} \cancel{-18kx^2e^{-3x}} \cancel{+9hx^3e^{-3x}} \cancel{+9kx^2e^{-3x}} = 6xe^{-3x} - 2e^{-3x},$$

we get

$$6hxe^{-3x} + 2ke^{-3x} = 6xe^{-3x} - 2e^{-3x},$$

that is the equalities $6h = 6, 2k = -2$ between the corresponding coefficients and then $h = 1, k = -1$.

The solution of the given differential equation is then

$$y(x) = Ae^{-3x} + Bxe^{-3x} + x^3e^{-3x} - x^2e^{-3x},$$

whose first derivative is

$$y'(x) = -3Ae^{-3x} + Be^{-3x} - 3Bxe^{-3x} + 3x^2e^{-3x} - 3x^3e^{-3x} - 2xe^{-3x} + 3x^2e^{-3x},$$

from which, by imposing the *initial conditions* $y(0) = -1, y'(0) = 1$ of the *Cauchy problem*, the system

$$\begin{cases} A & = & -1 \\ -3A + B & = & 1 \end{cases}$$

follows, having solution $A = -1, B = -2$. The solution of the given *Cauchy problem* is then

$$y(x) = -e^{-3x} - 2xe^{-3x} + x^3e^{-3x} - x^2e^{-3x}.$$

Exercise 3. The *Lagrangian function* $\mathcal{L}(x, y, z; \lambda)$ associated to the given optimization problem is

$$\mathcal{L}(x, y, z; \lambda) = x - y + z + \lambda\,(2x^2 + y^2 - xy - z^2 + z - 2/7),$$

from which the *first order conditions*

$$\begin{cases} 1 + 4\lambda x - \lambda y = 0 \\ -1 + 2\lambda y - \lambda x = 0 \\ 1 - 2\lambda z + \lambda = 0 \\ 2x^2 + y^2 - xy - z^2 + z - 2/7 = 0 \end{cases}$$

follow. If we solve the system consisting of the first two equations

$$\begin{cases} 4\lambda x - \lambda y = -1 \\ -\lambda x + 2\lambda y = 1 \end{cases}$$

with respect to x, y, we get

$$x = -\frac{1}{7\lambda} \qquad \text{and} \qquad y = \frac{3}{7\lambda},$$

whereas from the third equation, we get

$$z = \frac{\lambda + 1}{2\lambda},$$

that, inserted into the fourth equation, give

$$\frac{2}{49\lambda^2} + \frac{9}{49\lambda^2} + \frac{3}{49\lambda^2} - \frac{\lambda^2 + 2\lambda + 1}{4\lambda^2} + \frac{\lambda + 1}{2\lambda} - \frac{2}{7} = 0 \qquad \Longrightarrow \qquad \frac{1 - \lambda^2}{4\lambda^2} = 0,$$

where $\lambda \neq 0$ because $\lambda = 0$ can not be a *Lagrange's multiplier*. From $1 - \lambda^2 = 0$, we get $\lambda = \pm 1$ and then the *optimal points* $(x, y, z; \lambda)$ having coordinates

$$A = \left(-\frac{1}{7}, \frac{3}{7}, 1; 1\right) \qquad \text{and} \qquad B = \left(\frac{1}{7}, -\frac{3}{7}, 0; -1\right).$$

The *bordered hessian matrix* of this optimization problem is

$$\overline{H}(x,y,z;\lambda)=\begin{pmatrix} 0 & 4x-y & 2y-x & 1-2z \\ 4x-y & 4\lambda & -\lambda & 0 \\ 2y-x & -\lambda & 2\lambda & 0 \\ 1-2z & 0 & 0 & -2\lambda \end{pmatrix},$$

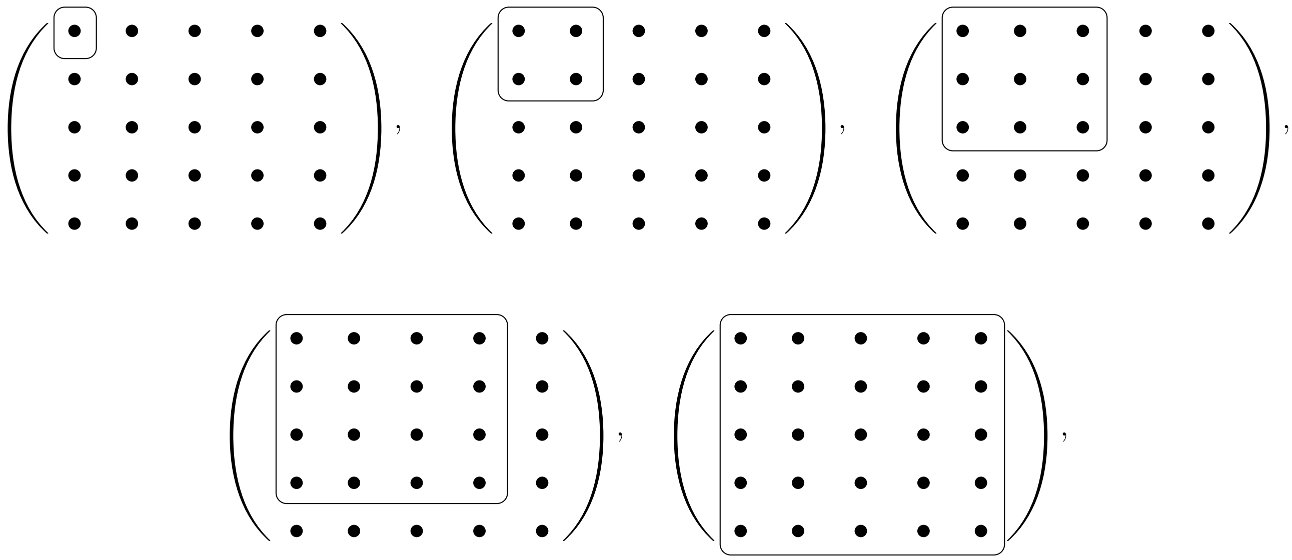
and we remind the general *second order conditions* based on the analysis of the *bordered hessian matrix*.

Given a square matrix $\overline{H}$ of order n and a positive integer number $k \leqslant n$, the *minor matrix* consisting of the first k rows and the first k columns of $\overline{H}$ is called *leading principal minor* of order k included in the matrix $\overline{H}$.

In order to fix the ideas, we consider for example a square matrix of order 5

$$\overline{H}=\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix},$$

in which we highlight all *leading principal minors*, from the order 1 until the highest possible order 5



and we denote by $\mathcal{H}_k$ the *determinant* of the *leading principal minor* of order k included in the matrix $\overline{H}$.

The general *second order conditions* based on the analysis of the *bordered hessian matrix* $\overline{H}$ now read in the following way. Given the *optimization problem* consisting of optimizing a function depending on n variables subject to $p < n$ constraints, we consider the *bordered hessian matrix* $\overline{H}(P)$ corresponding to the *optimization problem*, evaluated in an *optimal point* P determined by means of the *first order conditions*. We then have that

- if it yields

$$\begin{gathered} (-1)^{p+1}\mathcal{H}_{2p+1}(P)>0, \\ (-1)^{p+2}\mathcal{H}_{2p+2}(P)>0, \\ (-1)^{p+3}\mathcal{H}_{2p+3}(P)>0, \\ \vdots \\ (-1)^{n}\mathcal{H}_{n+p}(P)>0, \end{gathered} \tag{8a}$$

  the point P is the *maximum point*;

- if it yields

$$\begin{gathered} (-1)^{p}\mathcal{H}_{2p+1}(P)>0, \\ (-1)^{p}\mathcal{H}_{2p+2}(P)>0, \\ (-1)^{p}\mathcal{H}_{2p+3}(P)>0, \\ \vdots \\ (-1)^{p}\mathcal{H}_{n+p}(P)>0, \end{gathered} \tag{8b}$$

the point P is the *minimum point*.

It is important to point out that conditions (8) are *sufficient conditions*, only, and it is also possible that they do not hold. If conditions (8) do not hold, we have to conclude that the *nature* of the *optimal point* can not be determined by means of the *second order conditions* (8), and conditions of higher order are have to be studied.

In the exercise of the exam, we have the *bordered hessian matrices* evaluated in the two *optimal points* A, B

$$\overline{H}(A) = \begin{pmatrix} 0 & -1 & 1 & -1 \\ -1 & 4 & -1 & 0 \\ 1 & -1 & 2 & 0 \\ -1 & 0 & 0 & -2 \end{pmatrix} \qquad \text{and} \qquad \overline{H}(B) = \begin{pmatrix} 0 & 1 & -1 & 1 \\ 1 & -4 & 1 & 0 \\ -1 & 1 & -2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}.$$

Since we have $n = 3$ variables and $p = 1$ constraint, we have $2p + 1 = 3$ and $n + p = 4$, that is we have to compute the determinant of the *leading principal minors* of order 3 and of order 4 of the *bordered hessian matrices* $\overline{H}(A), \overline{H}(B)$ evaluated in the *optimal points*. By virtue of conditions (8a), we have that if it yields

$$\mathcal{H}_3(P) > 0 \qquad \text{and} \qquad \mathcal{H}_4(P) < 0,$$

the point P is the *maximum point*; if it yields

$$\mathcal{H}_3(P) < 0 \qquad \text{and} \qquad \mathcal{H}_4(P) < 0,$$

the point P is the *minimum point*.

Since we have

$$\det \overline{H}(A) = \det \overline{H}(B) = 1 > 0,$$

we conclude that the nature of the *optimal points* A, B can not be studied by means of the *second order conditions* at disposal.